Economics 204 Summer/Fall 2022
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

Do not turn the page until the exam begins.

1. (15) Define or state each of the following.
   (a) eigenvector of a linear transformation $T : X \to Y$ between vector spaces X and Y over the same field F
   (b) open set in a metric space (X, d)
   (c) Intermediate Value Theorem

2. (30) Let A and B be $n \times n$ matrices that commute, so $AB = BA$. Show that for every $k \in \mathbb{N}$ with $k \geq 2$, $A^k B = BA^k$ (where M^k is the product of k copies of the $n \times n$ matrix M).

   (**Hint:** use induction.)

3. (30) Let X and Y be vector spaces over the same field F, and let $T : X \to Y$ be a linear transformation. Let $V \subseteq X$ be linearly independent. Show that if T is one-to-one, then $T(V) \subseteq Y$ is linearly independent.

4. (30) Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. Let $A \subseteq X$. Show that $f(\bar{A}) \subseteq \overline{f(A)}$.

5. (30) Let $U \subseteq \mathbf{R}^n$ be open and $f : U \to \mathbf{R}$ be differentiable on U. Suppose for each $x \in U$ there exists $\varepsilon_x > 0$ and $M_x > 0$ such that $\|Df(y)\| \leq M_x$ for all $y \in B_{\varepsilon_x}(x)$.

   Suppose $C \subseteq U$ is convex and compact. Show that f is Lipschitz continuous on C. (That is, show that there exists $M > 0$ such that $\|f(x) - f(y)\| \leq M\|x - y\|$ for all $x, y \in C$.)

   (**Hint:** Show that there exists $M > 0$ such that $\|Df(z)\| \leq M$ for all $z \in C$.)

6. (30) Let (X, d_1) be a nonempty, complete metric space and $C \subseteq \mathbf{R}^n$ be a nonempty, compact, convex set. Consider the metric space $(X \times C, d)$, where $d : X \times C \to \mathbf{R}_+$ is the metric given by $d((x, y), (z, w)) = d_1(x, z) + d_2(y, w)$ for $(x, y), (z, w) \in X \times C$, where d_1 is the metric on X and d_2 denotes the standard metric in $\mathbf{R}^n$ (you can use without proof that d is a metric on $X \times C$.)

   Let $f : X \times C \to X \times C$, and write $f(x, y) = (f_1(x, y), f_2(x, y))$, where $f_1 : X \times C \to X$ and $f_2 : X \times C \to C$. Suppose f is Lipschitz continuous, and for each $y \in C$, $f_1(\cdot, y) : X \to X$ is a contraction on (X, d_1). Show that f has a fixed point.