Announcements

- PS 3 due today
  ~ solns posted tomorrow morning
- PS 4 posted
  - due Tues 8/9
- Last year's exam posted ~ Sunday

Econ 204 2023

Lecture 10

Outline

0. Eigenvectors + Diagonalization
1. Diagonalization of Real Symmetric Matrices
2. Application to Quadratic Forms
3. Linear Maps Between Normed Spaces

Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that λ is an eigenvalue of T if and only if λ is an eigenvalue for some matrix representation of T if and only if λ is an eigenvalue for every matrix representation of T.

Definition 2. *Let X be a vector space and $T \in L(X, X)$. We say that λ* $\in F$ *is an* eigenvalue *of T and $v \neq 0$ is an* eigenvector corresponding to λ *if $T(v) = \lambda v$.*

$Mtx_u(T)\ crd_u(v) = crd_u(T(v))$

$T(v) = T(v)$

Eigenvalues and Eigenvectors

Theorem 6 (Theorem 4 in Handout)**.** *Let X be a finite-dimensional vector space, and U a basis. Then λ is an eigenvalue of T if and only if λ is an eigenvalue of $Mtx_U(T)$. v is an eigenvector of T corresponding to λ if and only if $crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corresponding to λ.*

Proof. By the Commutative Diagram Theorem,

$$\exists\, v \neq 0 \text{ s.t. } T(v) = \lambda v \;\Leftrightarrow\; crd_U(T(v)) = crd_U(\lambda v) \qquad v \neq 0$$
$$\Leftrightarrow\; Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v))$$

□

$A = Mtx_u(T)$, $x = crd_u(v)$,

$\Leftrightarrow Ax = \lambda x$

Computing Eigenvalues and Eigenvectors

Suppose $\dim X = n$; let I be the $n \times n$ identity matrix. Given $T \in L(X, X)$, fix a basis U and let

$$A = Mtx_U(T)$$

Find the eigenvalues of T by computing the eigenvalues of A:

$$\begin{aligned} \exists v \neq 0 \text{ s.t. } Av = \lambda v \quad &\iff \quad (A - \lambda I)v = 0 \quad \text{for some } v \neq 0 \\ &\iff \quad (A - \lambda I) \text{ is not invertible} \\ &\iff \quad \det(A - \lambda I) = 0 \end{aligned}$$

We have the following facts:

- If $A \in \mathbf{R}_{n \times n}$,

$$f(\lambda) = \det(A - \lambda I)$$

is an n^{th} degree polynomial in λ with real coefficients; it is called the *characteristic polynomial* of A.

- f has n roots in $\mathbf{C}$, counting multiplicity:

$$f(\lambda) = (c_1 - \lambda)(c_2 - \lambda) \cdots (c_n - \lambda)$$ (may have $c_i = c_j$, $i \neq j$)

where $c_1, \ldots, c_n \in \mathbf{C}$ are the eigenvalues; the c_j's are not necessarily distinct. Notice that $f(\lambda) = 0$ if and only if $\lambda \in \{c_1, \ldots, c_n\}$, so the roots are the solutions of the equation $f(\lambda) = 0$.

- the roots that are not real come in conjugate pairs:

$$f(a+bi) = 0 \Leftrightarrow f(a-bi) = 0$$

- if $\lambda = c_j \in \mathbf{R}$, there is a corresponding eigenvector in $\mathbf{R}^n$.

- if $\lambda = c_j \notin \mathbf{R}$, the corresponding eigenvectors are in $\mathbf{C}^n \setminus \mathbf{R}^n$.

Diagonalization

Definition 3. *Suppose X is a finite-dimensional vector space with basis U. Given a linear transformation $T \in L(X, X)$, let*

$$A = Mtx_U(T)$$

We say that A can be diagonalized *if there is a basis W for X such that $Mtx_W(T)$ is a diagonal matrix, that is,*

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

So

$$\begin{aligned} A \text{ can be diagonalized} \quad &\iff \quad A \text{ is similar to a diagonal matrix} \\ &\iff \quad A = P^{-1}BP \text{ where } B \text{ is diagonal} \end{aligned}$$

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \lambda_n \\ 0 & \cdots & \cdots & \end{pmatrix}$$

Suppose there is a basis W such that

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$\Rightarrow$ $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $mtx_W(T)$ and T

Then the standard basis vectors of $\mathbf{R}^n$ are eigenvectors of $Mtx_W(T)$.

In general:

z_j is an eigenvector of T corresponding to λ_j $\iff$ $crd_W(z_j)$ is an eigenvector of $Mtx_W(T)$ corresponding to λ_j.

So an eigenvector of T corresponding to λ_j is w_j, since $crd_W(w_j) = e_j$, the j^{th} standard basis vector in $\mathbf{R}^n$.

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix}$$

$$\|$$

Thus $Mtx_W(T)$ is diagonal if and only if $W = \{w_1, \ldots, w_n\}$ where w_j is an eigenvector of T corresponding to λ_j for each j.

Then the action of T is clear: it stretches each basis element w_i by the factor λ_i.

Diagonalization

Theorem 7 (Thm. 6.7'). *Let X be an n-dimensional vector space, $T \in L(X, X)$, U any basis of X, and $A = Mtx_U(T)$. Then the following are equivalent:*

1. *A can be diagonalized*
2. *there is a basis W for X consisting of eigenvectors of T*
3. *there is a basis V for $\mathbf{R}^n$ consisting of eigenvectors of A*

Proof. Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout. $\square$

Diagonalization

Theorem 8 (Thm. 6.8'). *Let X be a vector space and $T \in L(X, X)$.*

1. *If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors $v_1, \ldots, v_m$, then $\{v_1, \ldots, v_m\}$ is linearly independent.*

2. *If $\dim X = n$ and T has n distinct eigenvalues, then X has a basis consisting of eigenvectors of T; consequently, if U is any basis of X, then $Mtx_U(T)$ is diagonalizable.*

Proof. This is an adaptation of the proof of Theorem 6.8 in de la Fuente. $\square$

How Might This Matter

$c_{t+1} = b_{11} c_t + b_{12} k_t$

$k_{t+1} = b_{21} c_t + b_{22} k_t \quad \forall t$

- Why does diagonalizability matter?

  Consider a two-dimensional linear difference equation:

$$\begin{pmatrix} c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t = 0, 1, 2, 3, \ldots$$

  given an initial condition c_0, k_0, or, setting

$$y_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix} \quad \forall t \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

  we can rewrite this more compactly as

$$y_{t+1} = B y_t \quad \forall t$$

  where $b_{ij} \in \mathbf{R}$ each i, j.

We want to find a solution y_t, $t = 1, 2, 3, \ldots$ given initial condition y_0. (Why?)

Such a dynamical system will arise for example as a characterization of the solution to a standard infinite-horizon optimal growth problem (202a, lecture 2).

$\boxed{\text{If } B \text{ is diagonalizable,}}$ this can be easily solved after a change of basis. If B is diagonalizable, choose an invertible 2×2 real matrix P such that

$$P^{-1}BP = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Then

$$\begin{aligned} y_{t+1} = By_t \quad \forall t \quad &\iff \quad P^{-1}y_{t+1} = P^{-1}By_t \quad \forall t \qquad \text{(mult. by } P^{-1}) \\ &\iff \quad P^{-1}y_{t+1} = (P^{-1}BP)(P^{-1}y_t) \quad \forall t \qquad PP^{-1} = I \\ &\iff \quad \bar{y}_{t+1} = D\bar{y}_t \quad \forall t \end{aligned}$$

$$= \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \bar{y}_t \quad , \bar{y}_t = P^{-1}y_t \ \forall t$$

$\iff \bar{y}_{i_{t+1}} = d_i \bar{y}_{i_t} \quad \forall t \ , \quad i = 1, 2$

where $\bar{y}_t = P^{-1} y_t \ \forall t$.

Since D is diagonal, after a change of basis to $\bar{y}_t$, we need to solve two **independent** linear univariate difference equations, which is easy:

$$\bar{y}_{it} = d_i^t \bar{y}_{i0} \quad \forall t$$

- Not all real $n \times n$ matrices are diagonalizable (not even all invertible $n \times n$ matrices are)...so can we identify some classes that are?
  - basis of eigenvectors ($\iff$)
  - n distinct eigenvalues ($\Rightarrow$)

- Some types of matrices appear more frequently than others – especially real symmetric $n \times n$ matrices (matrix representation of second derivatives of C^2 functions, quadratic forms...). e.g. second order conditions in optimization, checking concavity and convexity, Taylor series approximation of function

- Recall that an $n \times n$ real matrix A is *symmetric* if $a_{ij} = a_{ji}$ for all i, j, where a_{ij} is the $(i, j)^{\text{th}}$ entry of A.

Rest of this section: work in $\mathbb{R}^n$
- vector space
- norm
- inner product $\left(x \cdot y = \sum_{i=1}^{n} x_i y_i\right)$

Orthonormal Bases

Definition 1. *Let*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A basis $V = \{v_1, \ldots, v_n\}$ *of* $\mathbf{R}^n$ *is* orthonormal *if* $v_i \cdot v_j = \delta_{ij}$. $= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

In other words, a basis is orthonormal if each basis element has unit length ($\|v_i\|^2 = v_i \cdot v_i = 1 \ \forall i$), and distinct basis elements are perpendicular ($v_i \cdot v_j = 0$ for $i \neq j$).

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = (x \cdot x)^{1/2}$$

Orthonormal Bases

Remark: Suppose that $x = \sum_{j=1}^{n} \alpha_j v_j$ where $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbf{R}^n$. Then

$$\begin{aligned} x \cdot v_k &= \left(\sum_{j=1}^{n} \alpha_j v_j \right) \cdot v_k \\ &= \sum_{j=1}^{n} \alpha_j (v_j \cdot v_k) \\ &= \sum_{j=1}^{n} \alpha_j \delta_{jk} \quad = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \\ &= \alpha_k \end{aligned}$$

so

$$x = \sum_{j=1}^{n} (x \cdot v_j) v_j$$

Orthonormal Bases

Example: The standard basis of $\mathbf{R}^n$ is orthonormal.

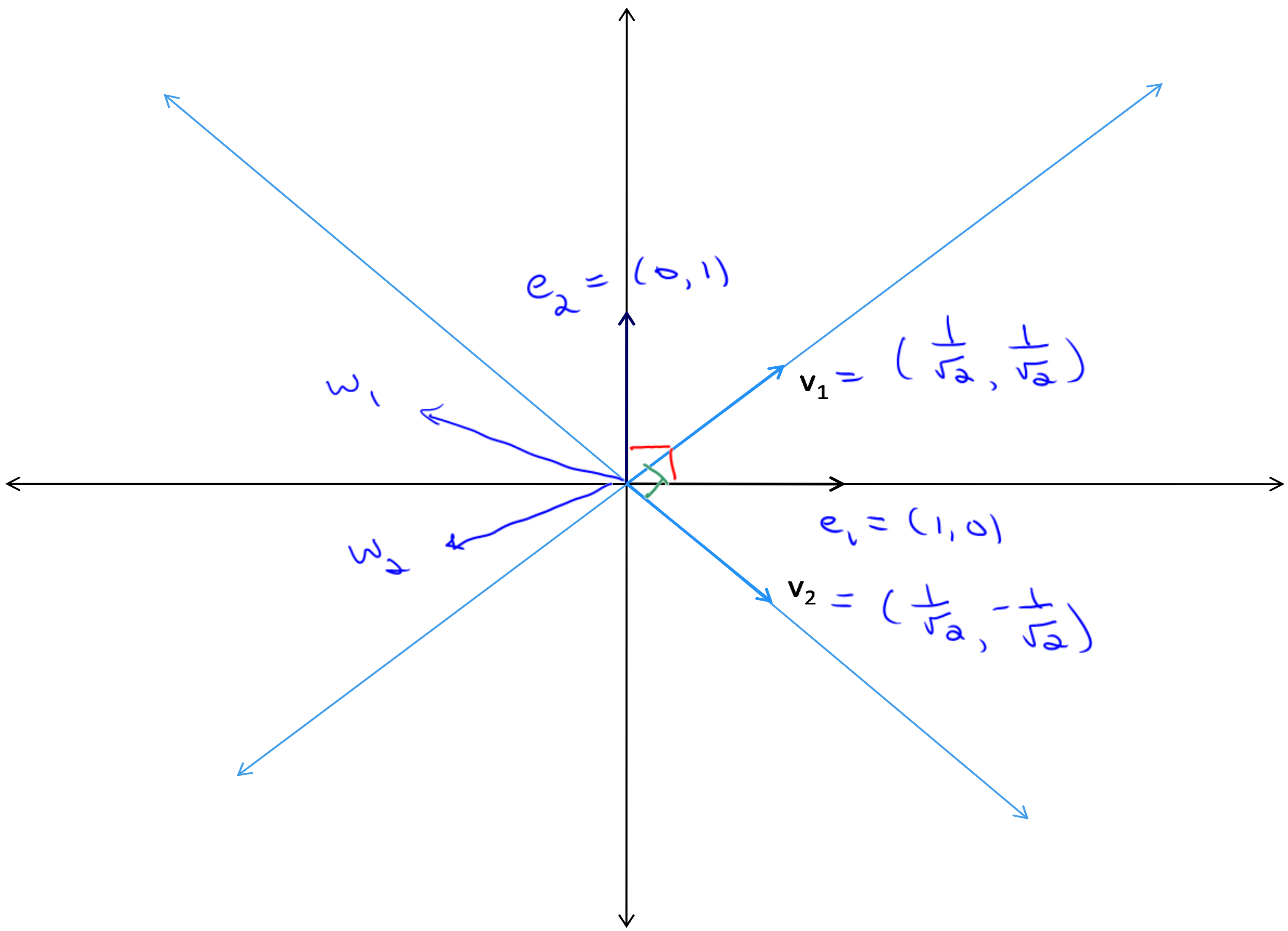
$e_i = (0, \ldots, 1, 0, \ldots, 0) \quad i = 1, \ldots, n$

(Why?)

e.g. $\mathbb{R}^2$: $e_1 = (1,0)$, $e_2 = (0,1)$

others? e.g. $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

also many bases that are <u>not</u> orthonormal



$e_2 = (0,1)$
$v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
w_1
$e_1 = (1,0)$
w_2
$v_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Unitary Matrices

Recall that for a real $n \times m$ matrix A, $A^\top$ denotes the transpose of A: the $(i,j)^{th}$ entry of $A^\top$ is the $(j,i)^{th}$ entry of A.

So the i^{th} row of $A^\top$ is the i^{th} column of A.

Definition 2. *A real $n \times n$ matrix A is* unitary *if $A^\top = A^{-1}$.*

$A^T A = I$

Notice that by definition every unitary matrix is invertible.

Unitary Matrices

Theorem 1. *A real $n \times n$ matrix A is unitary if and only if the columns of A are orthonormal.*

Proof. Let v_j denote the j^{th} column of A.

$$
\begin{aligned}
A^\top = A^{-1} \quad &\iff \quad A^\top A = I \quad = (\delta_{ij}) \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \\
&\iff \quad v_i \cdot v_j = \delta_{ij} \quad \forall i, j \\
&\iff \quad \{v_1, \ldots, v_n\} \text{ is orthonormal}
\end{aligned}
$$

□

$A = (v_1, \ldots, v_n)$ – $Mtx_{w,v}(id)$

$\uparrow \quad \cdots \quad \uparrow$

$crd_W(v_1) \quad crd_W(v_n)$

Unitary Matrices

$= \{v_1, \ldots, v_n\}$

If A is unitary, let V be the set of columns of A and W be the standard basis of $\mathbf{R}^n$. Since A is unitary, it is invertible, so V is a basis of $\mathbf{R}^n$. ($\{v_1, \ldots, v_n\}$ linearly independent)

$$A^\top = A^{-1} = Mtx_{V,W}(id)$$

– change of basis from W to V

W – standard basis

Since V is orthonormal, the transformation between bases W and V preserves all geometry, including lengths and angles.

Thm: Let C be an $n \times n$ real symmetric matrix.
Then C is diagonalizable. In addition,

$$C = P^{-1} D P$$

where D is a diagonal matrix and P is unitary.

Note: • The diagonal elements $\{\lambda_1, \ldots, \lambda_n\}$ of D are the eigenvalues of C

• C has orthonormal eigenvectors $\{v_1, \ldots, v_n\}$ that are a basis for $\mathbb{R}^n$.

Diagonalization of Real Symmetric Matrices

Theorem 2. *Let* $T \in L(\mathbf{R}^n, \mathbf{R}^n)$ *and* W *be the standard basis of* $\mathbf{R}^n$. *Suppose that* $Mtx_W(T)$ *is symmetric. Then the eigenvectors of* T *are all real, and there is an orthonormal basis* $V = \{v_1, \ldots, v_n\}$ *of* $\mathbf{R}^n$ *consisting of eigenvectors of* T, *so that* $Mtx_W(T)$ *is diagonalizable:*

$$Mtx_W(T) = Mtx_{W,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,W}(id)$$

where $Mtx_V T$ *is diagonal and the change of basis matrices* $Mtx_{V,W}(id)$ *and* $Mtx_{W,V}(id)$ *are unitary.*

The proof of the theorem requires a lengthy digression into the linear algebra of *complex* vector spaces. A brief outline is in the notes.

quadratic form: polynomial with all terms of degree 2

Quadratic Forms

Example: Let $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

write as $f(x) = x^T A x$, A symmetric

Let

$$A = \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix}$$

$$x^T A x = (x_1 \quad x_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

so A is symmetric and

$$\begin{aligned} x^\top A x = (x_1, x_2) \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= (x_1, x_2) \begin{pmatrix} \alpha x_1 + \frac{\beta}{2} x_2 \\ \frac{\beta}{2} x_1 + \gamma x_2 \end{pmatrix} \\ &= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 \\ &= f(x) \end{aligned}$$

Notice $f(0) = 0$.

Can we determine anything about $f(x)$ for $x \neq 0$?

e.g. $f(x) \geq 0 \ \forall x$?

easy if $\beta = 0$ ...

Quadratic Forms

general form:

Consider a quadratic form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j \tag{1}$$

Let

$$\alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ji}}{2} & \text{if } i > j \end{cases}$$

above diagonal

below diagonal

Let

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \text{ so } f(x) = x^\top A x$$

real symmetric

Quadratic Forms

A is symmetric, so let $V = \{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

$$\begin{aligned} \text{Then } A &= U^\top D U \quad = U^{-1} D U \\ \text{where } D &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \\ \text{and } U &= Mtx_{V,W}(id) \text{ is unitary} \end{aligned}$$

U^{-1}

The columns of $U^\top$ (the rows of U) are the coordinates of $v_1, \ldots, v_n$, expressed in terms of the standard basis W. Given $x \in \mathbf{R}^n$, recall

$$x = \sum_{i=1}^{n} \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$

Quadratic Forms

So $\quad x = \sum_{i=1}^{n} \gamma_i v_i$

$$
\begin{aligned}
f(\gamma) = f(x) &= f\left(\sum \gamma_i v_i\right) \\
&= \left(\sum \gamma_i v_i\right)^\top A \left(\sum \gamma_i v_i\right) \qquad = x^T A x \\
&= \left(\sum \gamma_i v_i\right)^\top U^\top D U \left(\sum \gamma_i v_i\right) \\
&= \left(U \sum \gamma_i v_i\right)^\top D \left(U \sum \gamma_i v_i\right) \qquad \left((EF)^T = F^T E^T\right) \\
&= \left(\sum \gamma_i U v_i\right)^\top D \left(\sum \gamma_i U v_i\right) \qquad (U \text{ linear}) \\
&= (\gamma_1, \ldots, \gamma_n) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \qquad \leftarrow \text{U is change of basis matrix from W to V} \Rightarrow U v_i = e_i \ \forall i \\
&= \sum \lambda_i \gamma_i^2
\end{aligned}
$$

eigenvalues of A

eigenvalues of A

Quadratic Forms

$f(x) = x^T A x$

This prove the following corollary of Theorem 2.

Corollary 1. *Consider the quadratic form (1).* Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$

1. *f has a global minimum at 0 if and only if $\lambda_i \geq 0$ for all i; the level sets of f are ellipsoids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$.*

2. *f has a global maximum at 0 if and only if $\lambda_i \leq 0$ for all i; the level sets of f are ellipsoids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$.*

3. *If* $\lambda_i < 0$ *for some* i *and* $\lambda_j > 0$ *for some* j*, then* f *has a saddle point at 0; the level sets of* f *are hyperboloids with principal axes aligned with the orthonormal eigenvectors* $v_1, \ldots, v_n$.

$\| f(x) - f(y) \| \leq M \| x - y \|$

Bounded Linear Maps

(over $\mathbb{R}$)

Definition 3. *Suppose* X, Y *are normed vector spaces and* $T \in L(X,Y)$*. We say* T *is* bounded *if*

$$\exists \beta \in \mathbf{R}_{+} \textit{ s.t. } \quad \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X$$

Note this implies that T is Lipschitz with constant β.

Why not

$\exists\, C \in \mathbb{R}$ s.t. $\| T(x) \| \leq C \quad \forall x \in X$?

$$T(\alpha x) = \alpha T(x) \qquad \forall \alpha \in \mathbb{R}$$

$$\Rightarrow \quad \| T(\alpha x) \| = |\alpha| \, \| T(x) \|$$

Bounded Linear Maps

Much more is true:

Theorem 3 (Thms. 4.1, 4.3)**.** *Let X and Y be normed vector spaces and $T \in L(X, Y)$. Then*

$$\begin{aligned} & T \text{ is continuous at some point } x_0 \in X \\ \iff \quad & T \text{ is continuous at every } x \in X \\ \iff \quad & T \text{ is uniformly continuous on } X \\ \iff \quad & T \text{ is Lipschitz} \\ \iff \quad & T \text{ is bounded} \end{aligned}$$

Proof. Suppose T is continuous at x_0. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\|z - x_0\| < \delta \Rightarrow \|T(z) - T(x_0)\| < \varepsilon$$

Now suppose x is any element of X. If $\|y-x\| < \delta$, let $z =$ $\underbrace{z = y - x + x_0}_{z - x_0 = y - x}$, so $\|z - x_0\| = \|y - x\| < \delta$.

$$
\begin{aligned}
&\|T(y) - T(x)\| \\
&\quad = \|T(y-x)\| && \text{(T linear)} \\
&\quad = \|T(y - x + x_0 - x_0))\| && = \|T(z - x_0)\| \\
&\quad = \|T(z) - T(x_0)\| \\
&\quad < \varepsilon
\end{aligned}
$$

which proves that T is continuous at every x, and uniformly continuous.

We claim that T is bounded if and only if T is continuous at 0. Suppose T is not bounded. Then

$$\exists \{x_n\} \text{ s.t. } \quad \|T(x_n)\| > n\|x_n\| \quad \forall n$$

Note that $x_n \neq 0$ $\forall n$. Let $\varepsilon = 1$. Fix $\delta > 0$ and choose n such that $\frac{1}{n} < \delta$. Let

$$
\begin{aligned}
x_n' &= \frac{x_n}{n\|x_n\|} = \frac{1}{n\|x_n\|}\, x_n \\
\|x_n'\| &= \frac{\|x_n\|}{n\|x_n\|} \\
&= \frac{1}{n} \\
\|x_n' - 0\| = \|x_n'\| &< \delta \\
\|T(x_n') - \underbrace{T(0)}_{0}\| &= \|T(x_n')\| \\
&= \frac{1}{n\|x_n\|}\|T(x_n)\| \qquad (\text{defn of } x_n' + T \text{ linear}) \\
&> \frac{n\|x_n\|}{n\|x_n\|} \qquad (\text{defn of } x_n) \\
&= 1 \\
&= \varepsilon
\end{aligned}
$$

Since this is true for every δ, T is not continuous at 0. Therefore, T continuous at 0 implies T is bounded. Now, suppose T is bounded, so find $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for every $x \in X$. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then

$$\begin{aligned}
\|x - 0\| < \delta \quad &\Rightarrow \quad \|x\| < \delta \\
&\Rightarrow \quad \|T(x) - \overset{=0}{T(0)}\| = \|T(x)\| < M\delta \qquad \text{(defn of } M\text{)} \\
&\Rightarrow \quad \|T(x) - T(0)\| < \varepsilon \quad = M\delta
\end{aligned}$$

so T is continuous at 0.

Thus, we have shown that continuity at some point x_0 implies uniform continuity, which implies continuity at every point, which implies T is continuous at 0, which implies that T is bounded, which implies that T is continuous at 0, which implies that T is continuous at some x_0, so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose T is bounded, with constant $M^{>0}$. Then

$$\begin{aligned}\|T(x) - T(y)\| &= \|T(x-y)\| \qquad \text{(T linear)} \\ &\le M\|x-y\|\end{aligned}$$

so T is Lipschitz with constant M; conversely, if T is Lipschitz with constant M, then T is bounded with constant M. So all the statements are equivalent. □

$$\forall x \in X: \ \|T(x) - T(0)\| = \|T(x)\| \le M\|x-0\| = M\|x\|$$

Bounded Linear Maps

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

Theorem 4 (Thm. 4.5)**.** *Let X and Y be normed vector spaces, with* dim $X = n$. *Every $T \in L(X, Y)$ is bounded.*
$(n \in \mathbb{N})$

Proof. See de la Fuente. □

Topological Isomorphism

Definition 4. *A* topological isomorphism *between normed vector spaces X and Y is a linear transformation $T \in L(X, Y)$ that is invertible (one-to-one, onto), continuous, and has a continuous inverse.*

Two normed vector spaces X and Y are topologically isomorphic *if there is a topological isomorphism $T : X \to Y$.*

T bounded :
$\exists \beta > 0$ s.t.
$\|T(x)\| \leq \beta \|x\| \; \forall x$

$\Rightarrow \dfrac{\|T(x)\|}{\|x\|} \leq \beta$

$\forall x \neq 0$

The Space $B(X,Y)$

Suppose X and Y are normed vector spaces. We define

$$B(X,Y) = \{T \in L(X,Y) : T \text{ is bounded}\}$$

Define :

$$\|T\|_{B(X,Y)} = \sup\left\{\frac{\|T(x)\|_Y}{\|x\|_X} : x \in X, x \neq 0\right\} = \sup\{\|T(x)\|_Y : \|x\|_X = 1\}$$

* note : $\Rightarrow \|T(x)\| \leq \|T\| \|x\| \; \forall x \in X$ by definition

We skip the proofs of the rest of these results – read dIF.

The Space $B(X, Y)$

Theorem 5 (Thm. 4.8)**.** *Let* X, Y *be normed vector spaces. Then*

$$\Big(B(X, Y), \|\cdot\|_{B(X,Y)}\Big)$$

is a normed vector space.

The Space $B(\mathbf{R}^n, \mathbf{R}^m)$

representation

Theorem 6 (Thm. 4.9). *Let* $T \in L(\mathbf{R}^n, \mathbf{R}^m)$ *(*$= B(\mathbf{R}^n, \mathbf{R}^m)$*) with matrix* $A = (a_{ij})$ *with respect to the standard bases. Let*

$$M = \max\{|a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then

$$M \leq \|T\| \leq M\sqrt{mn}$$

.

Compositions

Theorem 7 (Thm. 4.10)**.** *Let* $R \in L(\mathbf{R}^m, \mathbf{R}^n)$ *and* $S \in L(\mathbf{R}^n, \mathbf{R}^p)$. *Then*

$$\|S \circ R\| \leq \|S\|\|R\|$$

Invertibility

Define $\Omega(\mathbf{R}^n) = \{T \in L(\mathbf{R}^n, \mathbf{R}^n) : T \text{ is invertible}\}$

Theorem 8 (Thm. 4.11'). *Suppose $T \in L(\mathbf{R}^n, \mathbf{R}^n)$ and E is the standard basis of $\mathbf{R}^n$. Then*

$$\begin{aligned}
&T \text{ is invertible} \\
\iff\quad &\ker T = \{0\} \quad \Big\} \text{ nice exercise} \\
\iff\quad &\det\left(Mtx_E(T)\right) \neq 0 \\
\iff\quad &\det\Big(Mtx_{V,V}(T)\Big) \neq 0 \text{ for every basis } V \\
\iff\quad &\det\Big(Mtx_{V,W}(T)\Big) \neq 0 \text{ for every pair of bases } V, W
\end{aligned}$$

Invertibility

Theorem 9 (Thm. 4.12)**.** *If* $S, T \in \Omega(\mathbf{R}^n)$*, then* $S \circ T \in \Omega(\mathbf{R}^n)$ *and*

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}$$

Invertibility

Theorem 10 (Thm. 4.14). *Let* $S, T \in L(\mathbf{R}^n, \mathbf{R}^n)$*. If* T *is invertible and*

$$\|T - S\| < \frac{1}{\|T^{-1}\|}$$

then S *is invertible. In particular,* $\Omega(\mathbf{R}^n)$ *is open in* $L(\mathbf{R}^n, \mathbf{R}^n) = B(\mathbf{R}^n, \mathbf{R}^n)$*.*

Theorem 11 (Thm. 4.15). *The function* $(\cdot)^{-1} : \Omega(\mathbf{R}^n) \to \Omega(\mathbf{R}^n)$ *that assigns* T^{-1} *to each* $T \in \Omega(\mathbf{R}^n)$ *is continuous.*

Quadratic Forms

The equation for a level set of f is

$$\{x \in \mathbb{R}^n : f(x) = C\} = \left\{\gamma \in \mathbf{R}^n : \sum_{i=1}^{n} \lambda_i \gamma_i^2 = C\right\} \qquad C \in \mathbb{R}$$

- $\boxed{\text{If } \lambda_i \geq 0 \text{ for all } i,}$ the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal axis along v_i is $\sqrt{C/\lambda_i}$ if $C \geq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C < 0$.

  $\Rightarrow f$ has global min at 0, $f(x) \geq 0 \ \forall x$

- $\boxed{\text{If } \lambda_i \leq 0 \text{ for all } i,}$ the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal

  $\Rightarrow f$ has global max at 0, $f(x) \leq 0 \ \forall x$

axis along v_i is $\sqrt{C/\lambda_i}$ if $C \leq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C > 0$.

- $\boxed{\text{If } \lambda_i > 0 \text{ for some } i \text{ and } \lambda_j < 0 \text{ for some } j,}$ the level set is a hyperboloid. For example, suppose $n = 2$, $\lambda_1 > 0$, $\lambda_2 < 0$. The equation is

$$
\begin{aligned}
C &= \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2 \\
&= \left(\sqrt{\lambda_1}\gamma_1 + \sqrt{|\lambda_2|}\gamma_2\right)\left(\sqrt{\lambda_1}\gamma_1 - \sqrt{|\lambda_2|}\gamma_2\right)
\end{aligned}
$$

$\Rightarrow$ f has a saddle point at 0

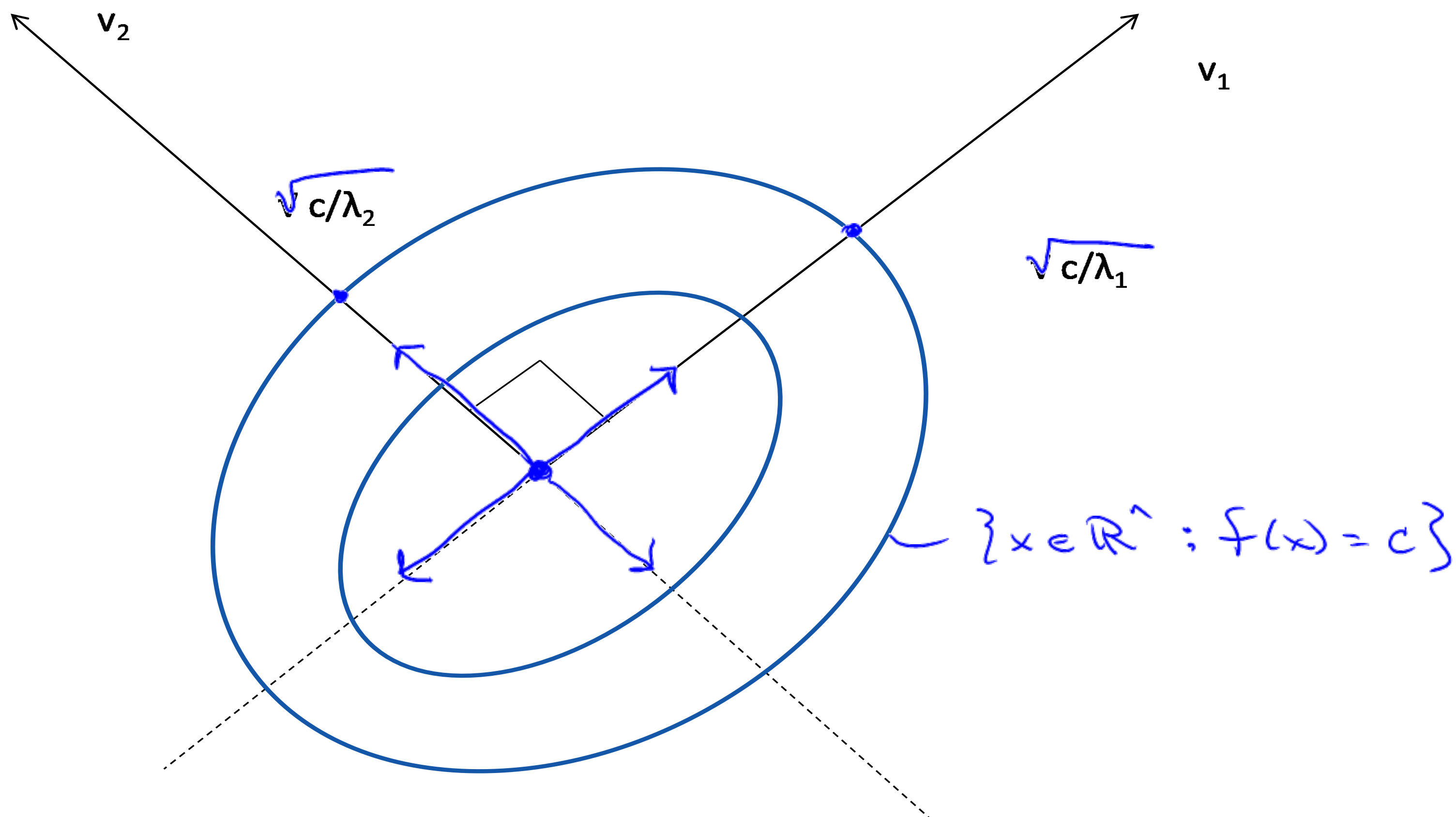
min with respect to v_i

max with respect to v_j

This is a hyperbola with asymptotes

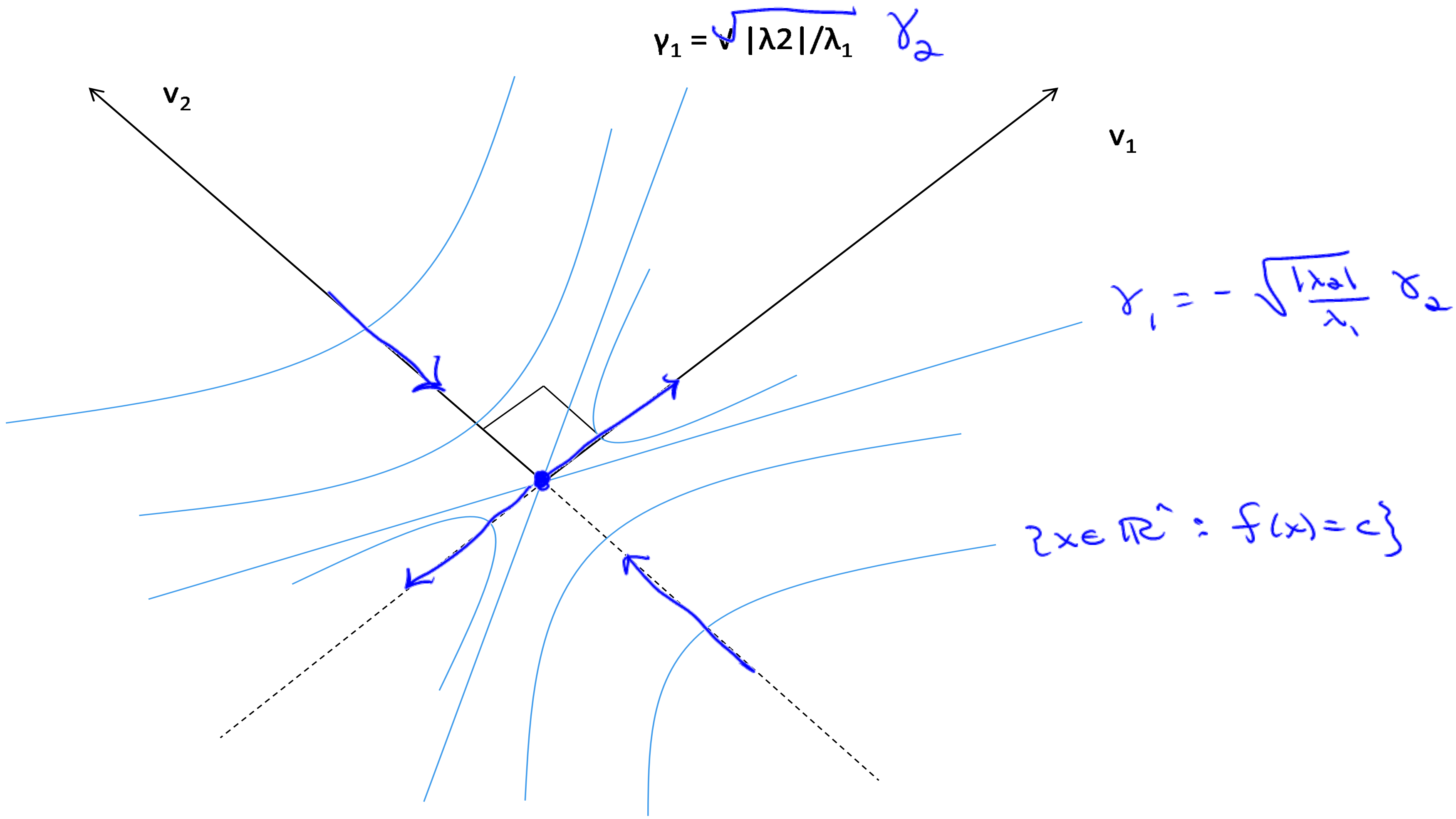
$$\begin{aligned} 0 &= \sqrt{\lambda_1}\gamma_1 + \sqrt{|\lambda_2|}\gamma_2 \\ \Rightarrow \gamma_1 &= -\sqrt{\frac{|\lambda_2|}{\lambda_1}}\gamma_2 \\ 0 &= \left(\sqrt{\lambda_1}\gamma_1 - \sqrt{|\lambda_2|}\gamma_2\right) \\ \Rightarrow \gamma_1 &= \sqrt{\frac{|\lambda_2|}{\lambda_1}}\gamma_2 \end{aligned}$$

$\lambda_1 > 0, \lambda_2 > 0$



f has a global min at 0

$\lambda_1 > 0, \lambda_2 < 0$



f has a saddle point at 0