Econ 204 2023

Lecture 13

Outline

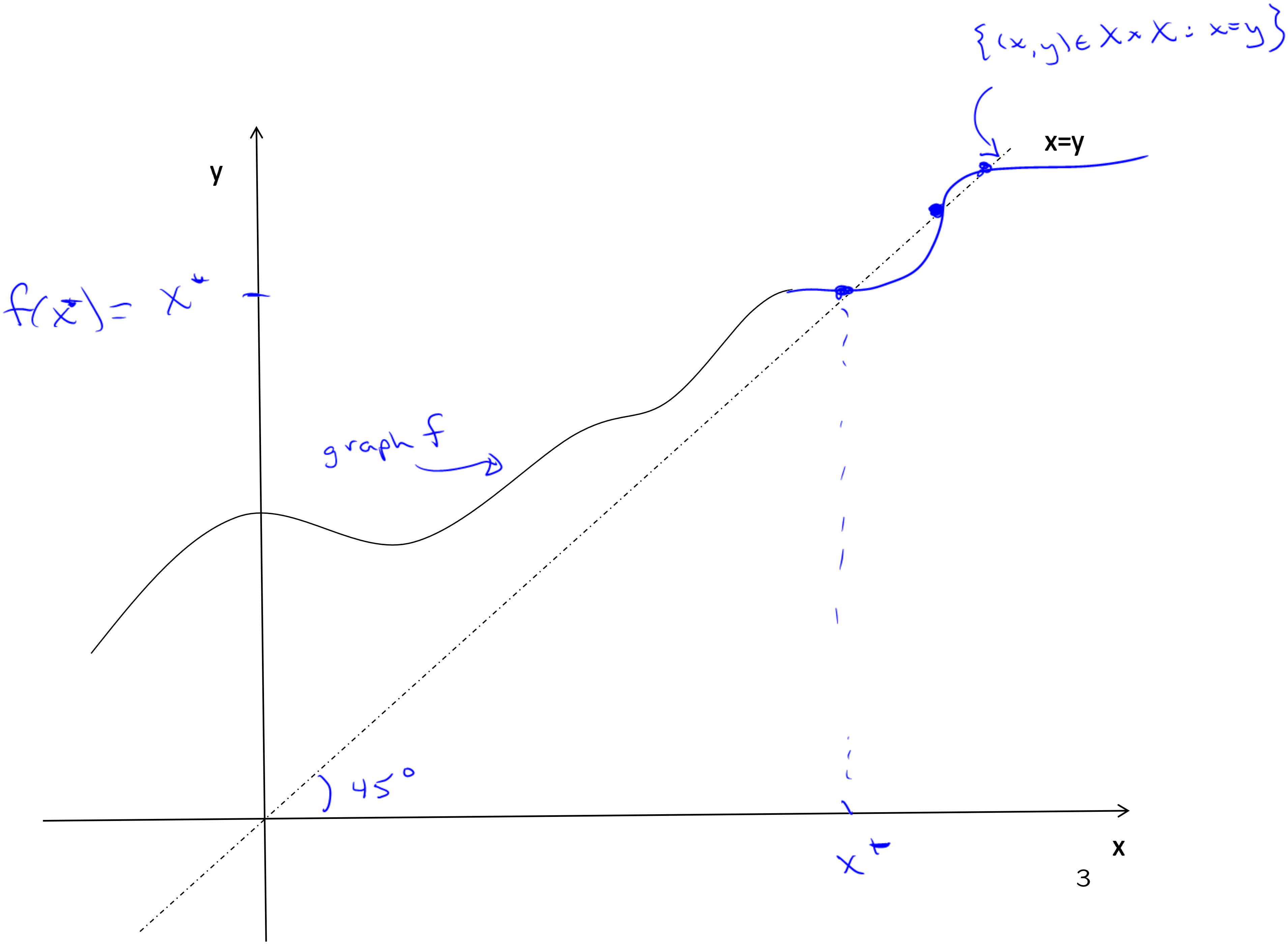
1. Fixed Points for Functions
2. Brouwer's Fixed Point Theorem
3. Fixed Points for Correspondences
4. Kakutani's Fixed Point Theorem
5. Separating Hyperplane Theorems

Recall :

Fixed Points for Functions

Definition 1. *Let* X *be a nonempty set and* $f : X \to X$*. A point* $x^* \in X$ *is a* fixed point *of* f *if* $f(x^*) = x^*$*.*

x^* is a fixed point of f if it is "fixed" by the map f.



{(x,y)∈X×X : x=y}
x=y
y
f(x*)= x*
graph f
45°
x
x*

$\exists$ a, b with $a < b$ s.t. $f : [a, b] \to [a, b]$

Fixed Points for Functions

Examples:

1. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x$. Then $x = 0$ is a fixed point of f (and is the unique fixed point of f).

   $f(x) = 2x = x \iff x = 0$

2. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x$. Then every point in $\mathbf{R}$ is a fixed point of f (in particular, fixed points need not be unique).

   $X = [0, 1]$ $f : X \to \mathbb{R}$

3. Let $X = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x + 1$. Then f has no fixed points.

   $f(x) = x + 1 \neq x \quad \forall x \in \mathbb{R}$

4. Let $X = [0,2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}(x+1)$. Then

$$\begin{aligned} f(x) &= \frac{1}{2}(x+1) = x \\ &\iff x + 1 = 2x \\ &\iff x = 1 \end{aligned}$$

So $x = 1$ is the unique fixed point of f. Notice that f is a contraction (why?), so we already knew that f must have a unique fixed point on $\mathbf{R}$ from the Contraction Mapping Theorem.

5. Let $X = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $f : X \to X$ be given by $f(x) = 1 - x$. Then f has no fixed points.

$$\begin{aligned} f(x) &= 1 - x = x \\ &\iff 1 = 2x \\ &\iff x = \tfrac{1}{2} \notin [0, \tfrac{1}{4}] \cup [\tfrac{3}{4}, 1] \end{aligned}$$

6. Let $X = [-2, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}x^2$. Then f has two fixed points, $x = 0$ and $x = 2$. If instead $X' = (0, 2)$, then $f : X' \to X'$ but f has no fixed points on X'.

7. Let $X = \{1, 2, 3\}$ and $f : X \to X$ be given by $f(1) = 2, f(2) = 3, f(3) = 1$ (so f is a permutation of X). Then f has no fixed points.

8. Let $X = [0, 2]$ and $f : X \to X$ be given by

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

$x \neq x+1 \quad \forall x \in \mathbb{R}$

$x \neq x-1 \quad \forall x \in \mathbb{R}$

Then f has no fixed points.

A Simple Fixed Point Theorem

Theorem 1. *Let $X = [a,b]$ for $a, b \in \mathbf{R}$ with $a < b$ and let $f : X \to X$ be continuous. Then f has a fixed point.*

Proof. Let $g : [a,b] \to \mathbf{R}$ be given by

$$g(x) = f(x) - x$$

$g(x) = 0 \iff x$ is a fixed point of f

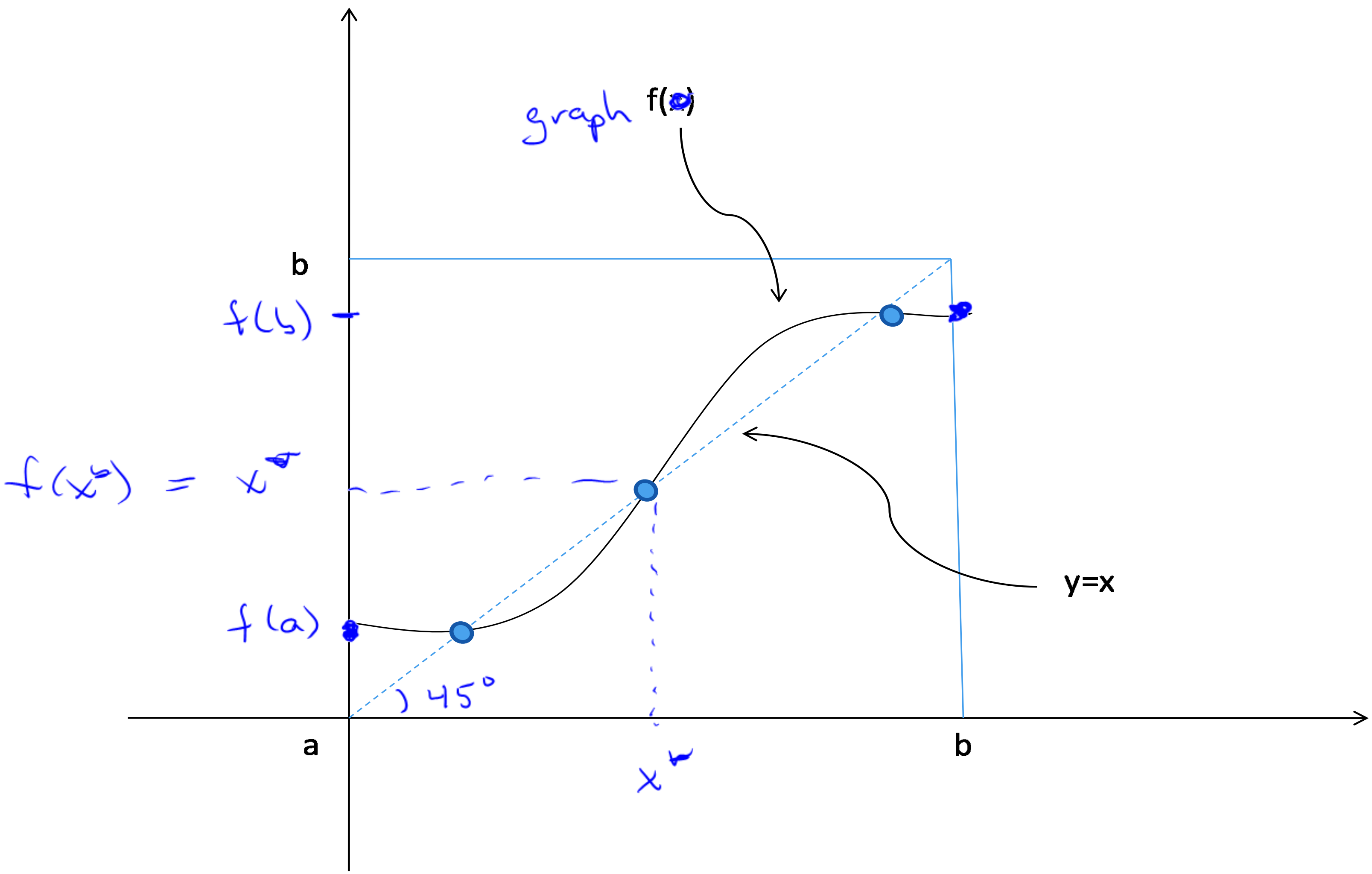
If either $f(a) = a$ or $f(b) = b$, we're done. So assume $f(a) > a$ and $f(b) < b$. Then

$(f(a) \in [a,b])$

$(f(b) \in [a,b])$

$$\begin{aligned} g(a) &= f(a) - a > 0 \\ g(b) &= f(b) - b < 0 \end{aligned}$$

g is continuous, so by the Intermediate Value Theorem, $\exists x^* \in (a,b)$ such that $g(x^*) = 0$, that is, such that $f(x^*) = x^*$. $\square$


graph f(x)
b
f(b)
f(x*) = x*
y=x
f(a)
45°
a
b
x*

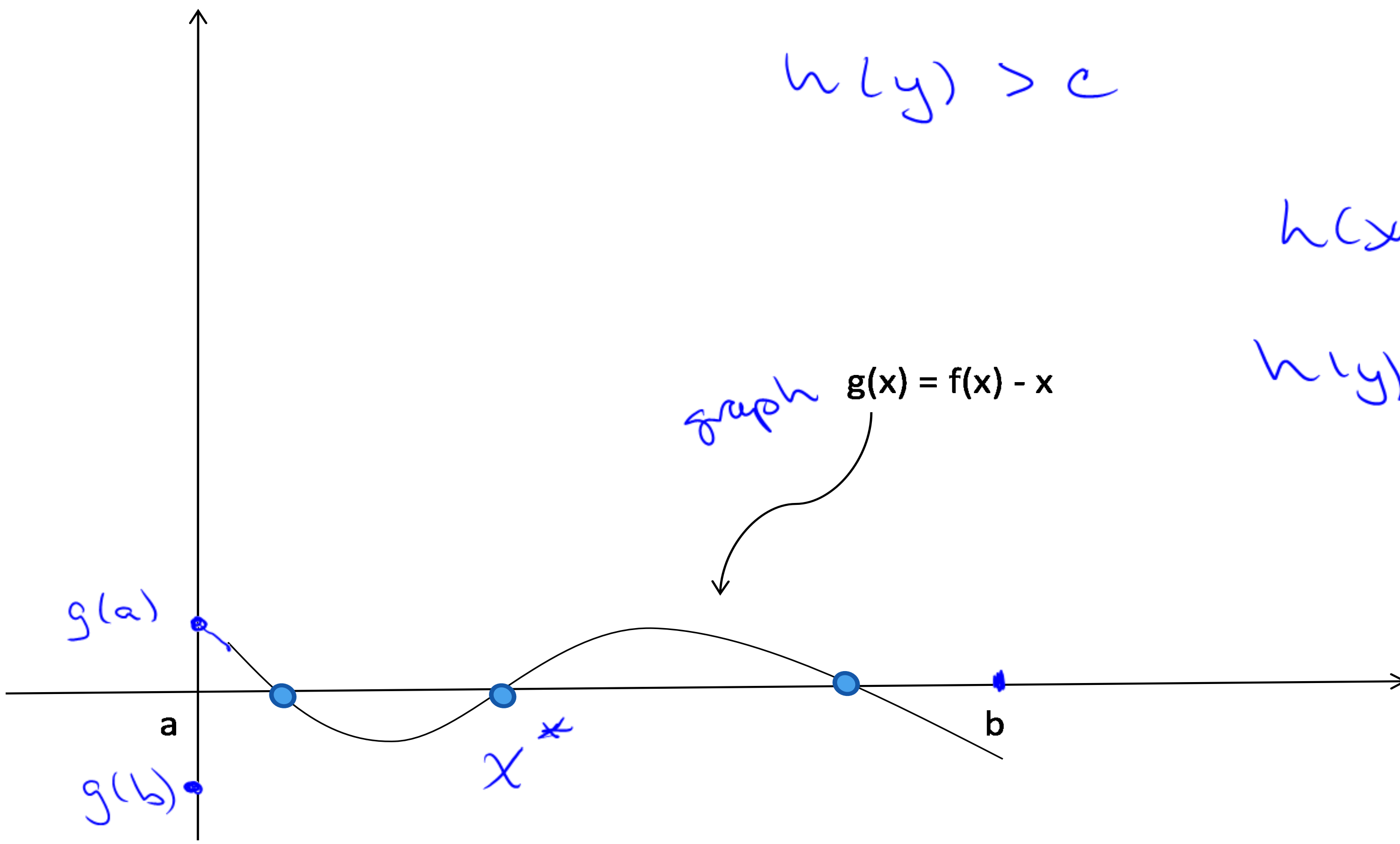
$s < c < t$

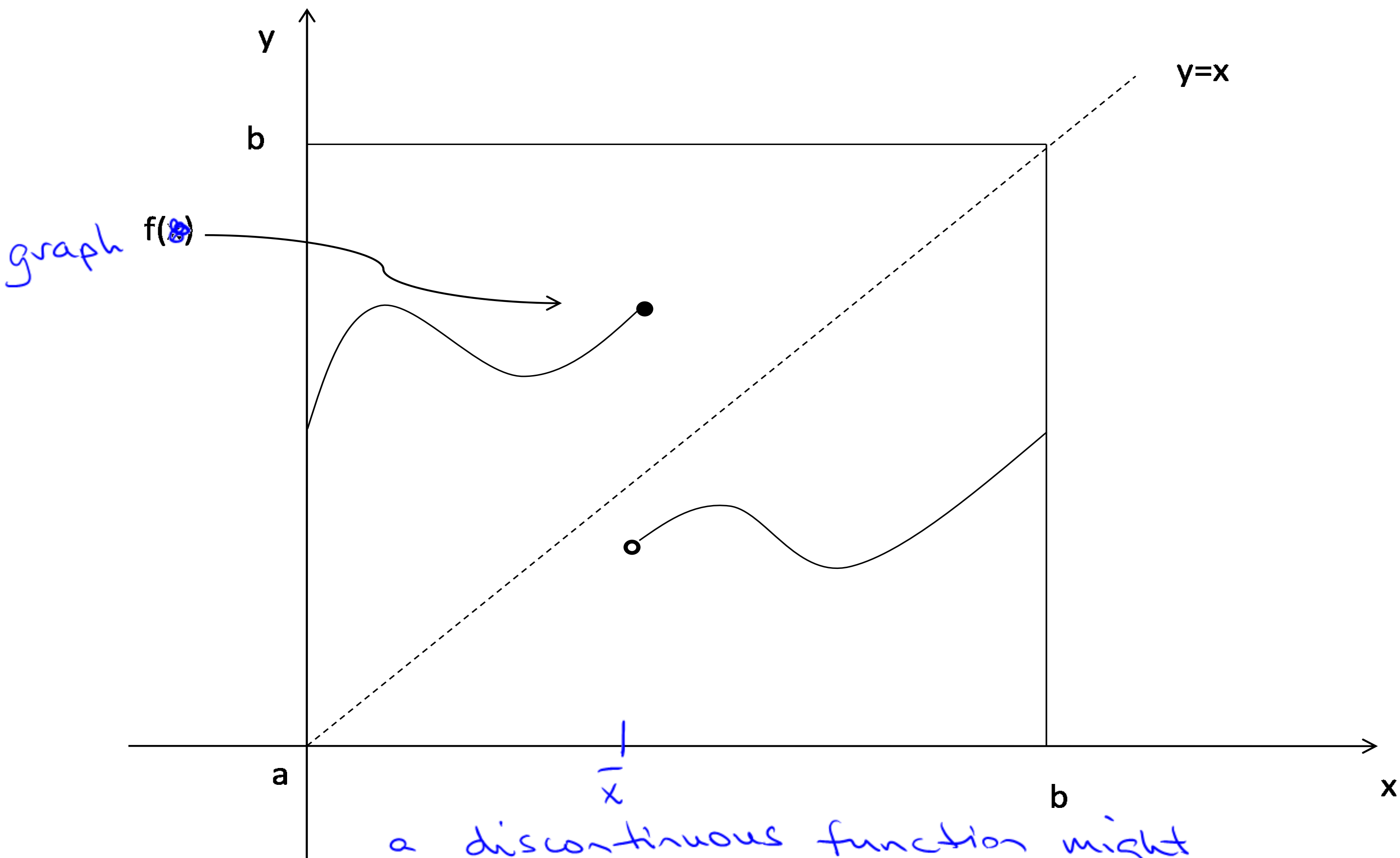
$h(x) < c$

$h(y) > c$

$h(x) > x$

$h(y) < y$





a discontinuous function might have no fixed points

$X \subseteq \mathbb{R}^n$ $\qquad$ $f: X \to \mathbb{R}^n$

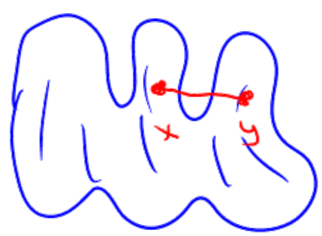
Brouwer's Fixed Point Theorem

Theorem 2 (Thm. 3.2. Brouwer's Fixed Point Theorem)**.** *Let $X \subseteq \mathbf{R}^n$ be nonempty, compact, and convex, and let $f : X \to X$ be continuous. Then f has a fixed point.*

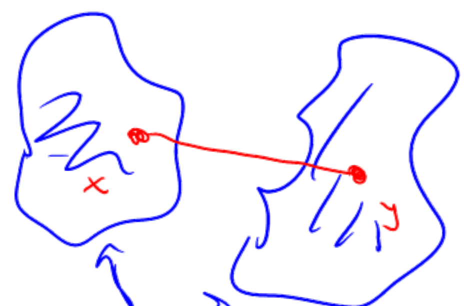
$X \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall x, y \in X$ $\forall \alpha \in [0,1]$

$\alpha x + (1-\alpha) y \in X$

X convex



D <u>not</u> convex



B not convex

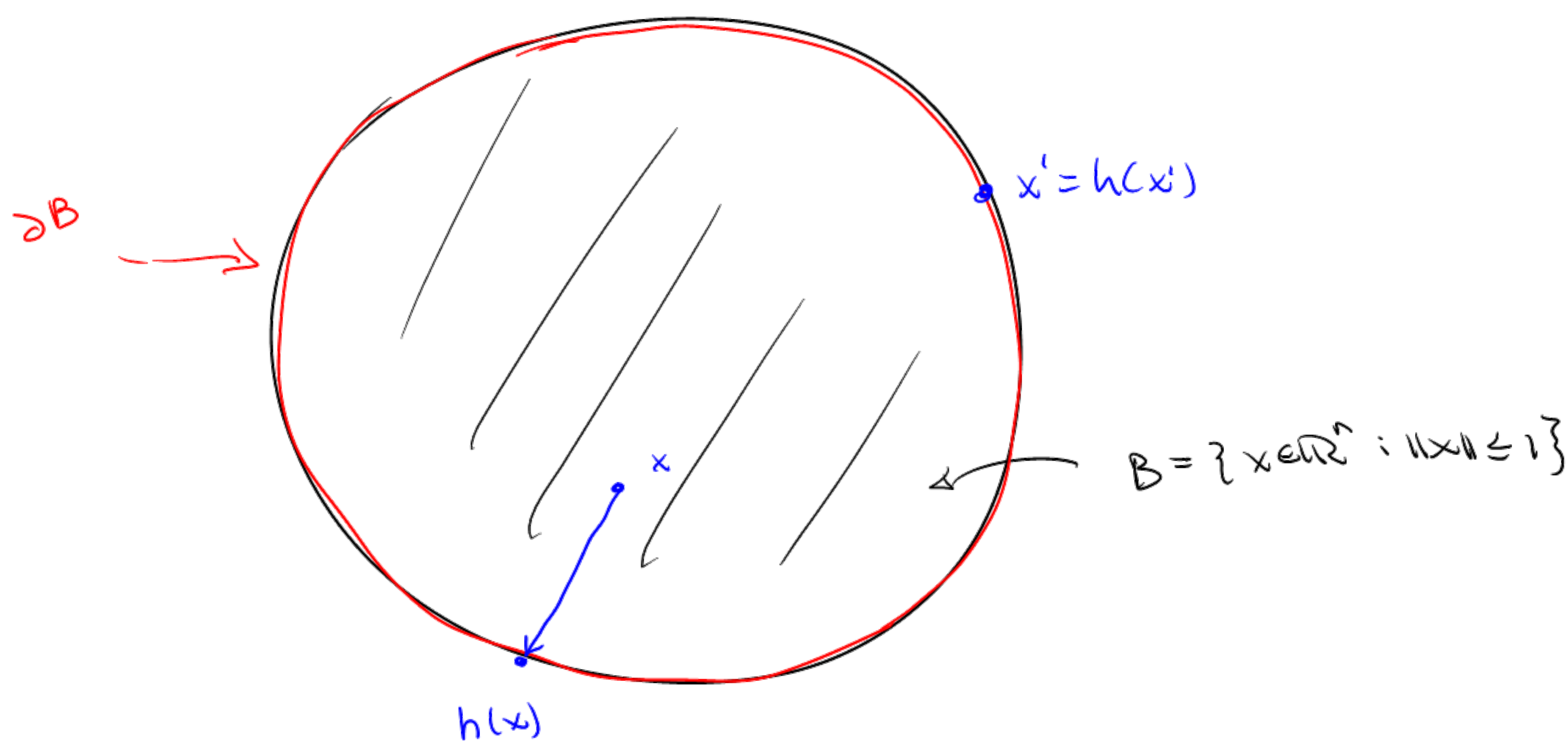
Sketch of Proof of Brouwer

Consider the case when the set X is the closed unit ball in $\mathbf{R}^n$, i.e. $X = B_1[0] = B = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$. Let $f : B \to B$ be a continuous function. Recall that ∂B denotes the boundary of B, so $\partial B = \{x \in \mathbf{R}^n : \|x\| = 1\}$.

Fact: Let B be the unit ball in $\mathbf{R}^n$. Then there is no continuous function $h : B \to \partial B$ such that $h(x') = x'$ for every $x' \in \partial B$.

See J. Franklin, Methods of Mathematical Economics, for an elementary (but long) proof.

(also Y. Kannai, Am. Math. Monthly, April 1981, pp. 264-268.)



$\nexists\ h : B \to \partial B$ continuous such that

$x' = h(x') \quad \forall x' \in \partial B$

Now to establish Brouwer's theorem, suppose, by way of contradiction, that f has no fixed points in B. Thus for every $x \in B$, $x \neq f(x)$.

Since $x \neq f(x)$ for every x, we can carry out the following construction. For each $x \in B$, construct the line segment originating at $f(x)$ and going through x. Let $g(x)$ denote the intersection of this line segment with ∂B.

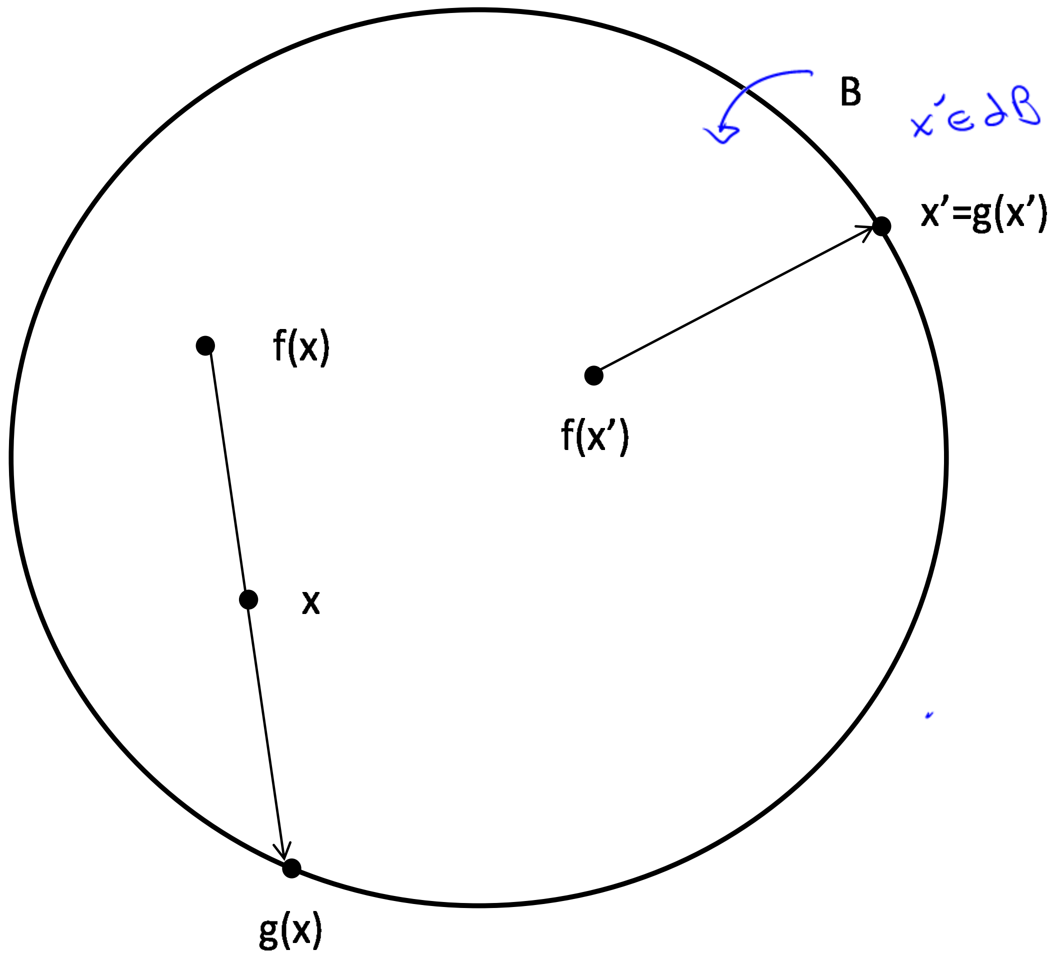
This construction is well-defined, and gives a continuous function $g : B \to \partial B$. Furthermore, if $x' \in \partial B$, then $x' = g(x')$. That is, $g|_{\partial B} = \mathrm{id}_{\partial B}$. Since there are no such functions by the fact above, we have a contradiction. Therefore there exists $x^* \in B$ such that $f(x^*) = x^*$, that is, f has a fixed point in B.

$g(x) = x + tu$

where $u = \dfrac{x - f(x)}{\| x - f(x) \|}$

$t = -x \cdot u + \sqrt{1 - x \cdot x + (x \cdot u)^2}$

$x' \in \partial B \Rightarrow$



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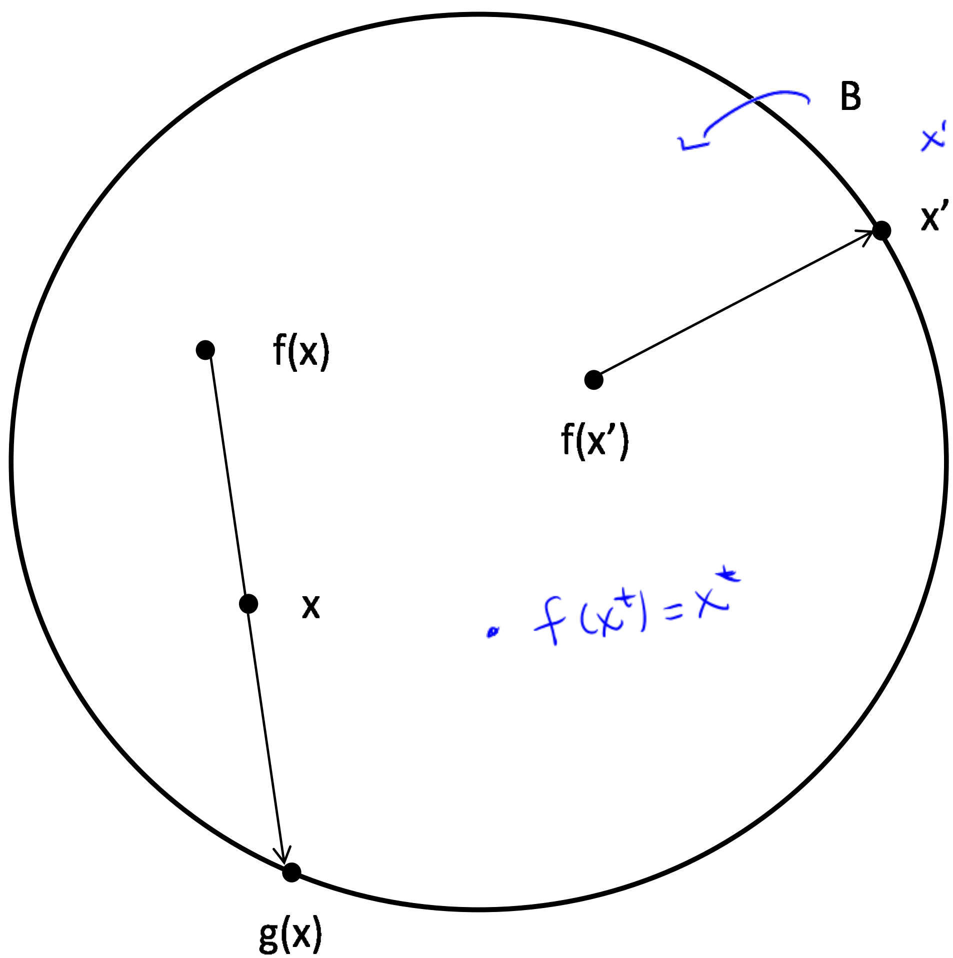


$x' \in \partial B \Rightarrow$

$\cdot \; f(x^*) = x^*$

$x \neq f(x) \;\; \forall x$

$\Rightarrow \; g : B \to \partial B$

is well-defined

and continuous

$\Rightarrow\!\Leftarrow$

Fixed Points for Correspondences

Definition 2. *Let X be nonempty and $\Psi : X \to 2^X$ be a correspondence. A point $x^* \in X$ is a* fixed point *of Ψ if $x^* \in \Psi(x^*)$.*

Note here that we do *not* require $\Psi(x^*) = \{x^*\}$, that is Ψ need not be single-valued at x^*. So x^* can be a fixed point of Ψ but there may be other elements of $\Psi(x^*)$ different from x^*.

Examples:

1. Let $X = [0,4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} [x+1, x+2] & \text{if } x < 2 \\ [0,4] & \text{if } x = 2 \\ [x-2, x-1] & \text{if } x > 2 \end{cases}$$

Then $x = 2$ is the unique fixed point of Ψ.

$2 \in \Psi(2) = [0,4]$

$x \notin [x+1, x+2]$

$x \notin [x-2, x-1]$

$\Rightarrow x \notin \Psi(x)$

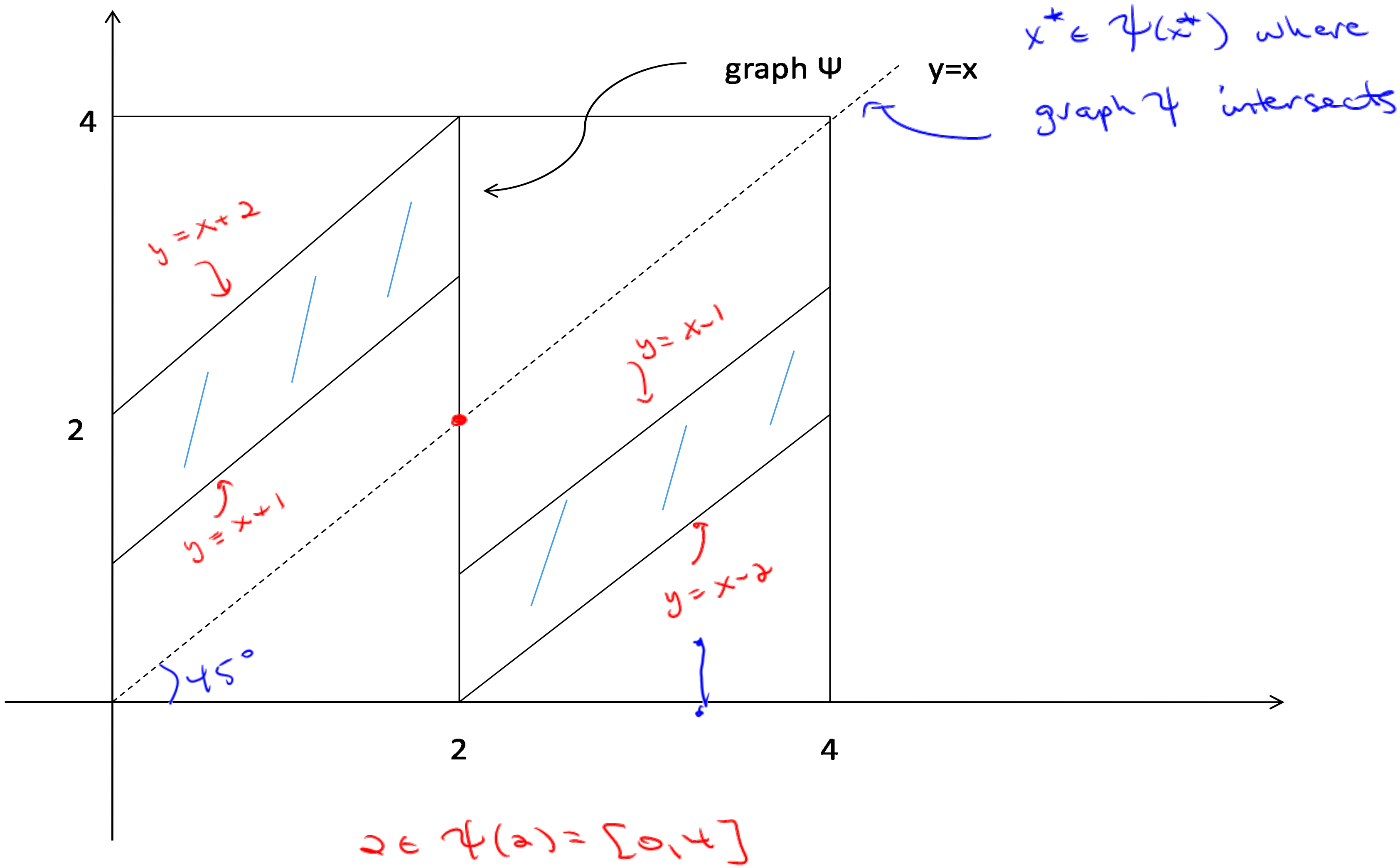
$\forall x \neq 2$

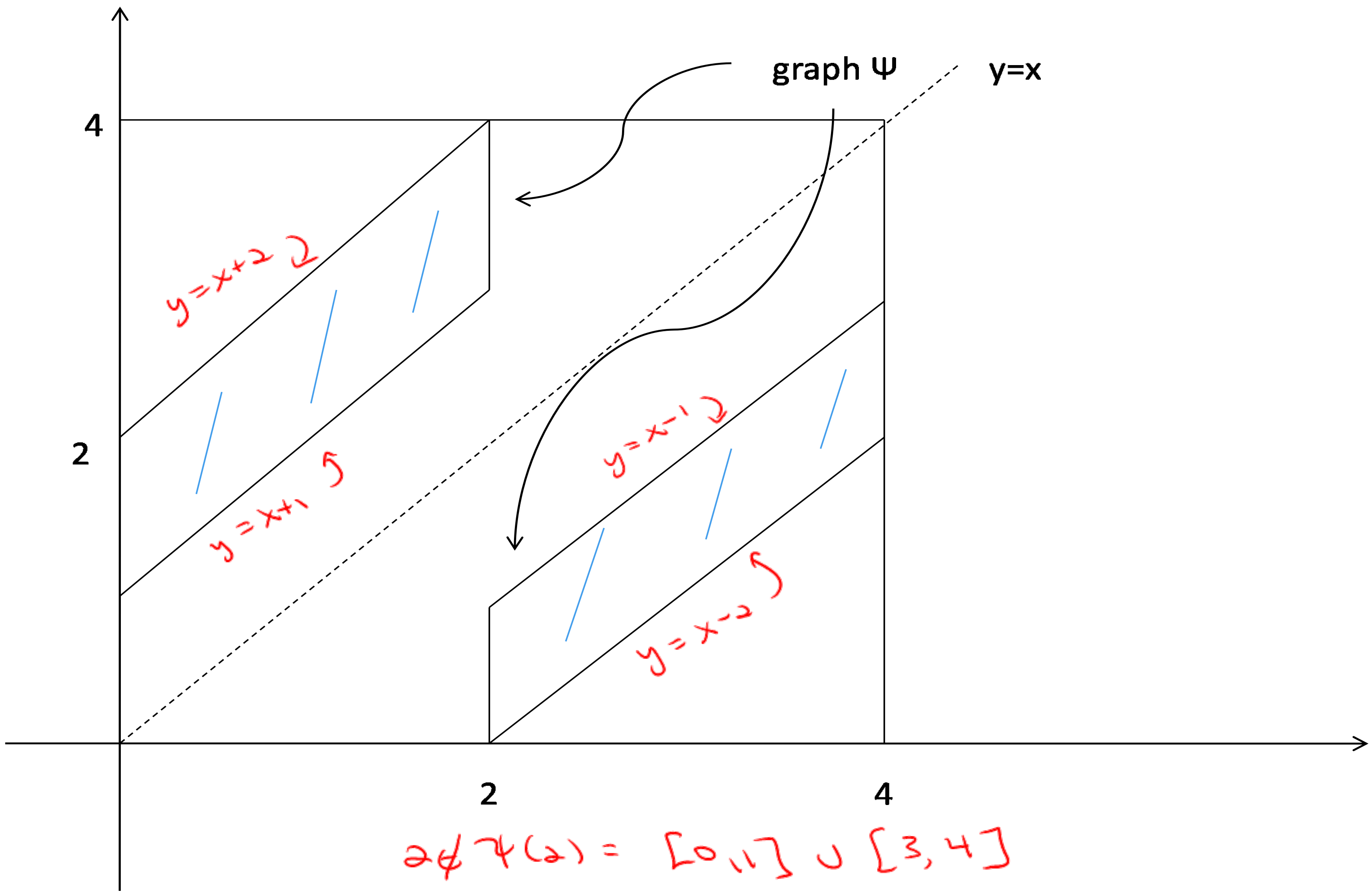
2. Let $X = [0,4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} [x+1, x+2] & \text{if } x < 2 \\ [0,1] \cup [3,4] & \text{if } x = 2 \\ [x-2, x-1] & \text{if } x > 2 \end{cases}$$

Then Ψ has no fixed points.

$2 \notin \Psi(2) = [0,1] \cup [3,4]$


graph Ψ
y=x
$x^* \in \Psi(x^*)$ where graph Ψ intersects
4
2
2
4
y = x+2
y = x+1
y = x-1
y = x-2
45°
$2 \in \Psi(2) = [0,4]$



$2 \notin \Psi(2) = [0,1] \cup [3,4]$

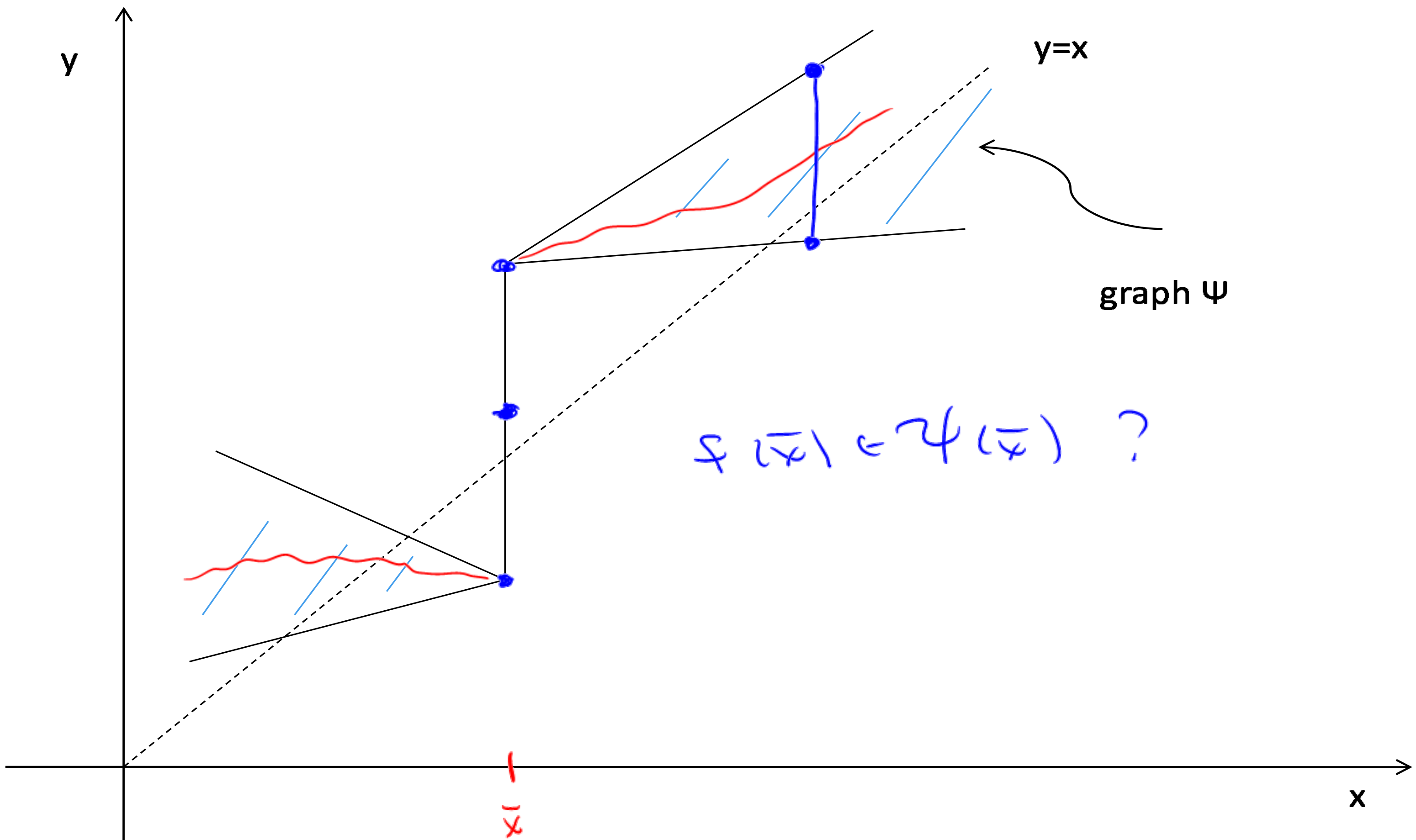
Note: Ψ is uhc in <u>both</u> cases

$\Psi(x) \subseteq X$ is nonempty, convex, compact
$\forall x \in X$

Kakutani's Fixed Point Theorem

Theorem 3. (Thm. 3.4'. Kakutani's Fixed Point Theorem) *Let $X \subseteq \mathbf{R}^n$ be a non-empty, compact, convex set and $\Psi : X \to 2^X$ be an upper hemi-continuous correspondence with non-empty, convex, compact values. Then Ψ has a fixed point in X.*

Proof. **(sketch)** Here, the idea is to use Brouwer's theorem after appropriately approximating the correspondence with a function. The catch is that there won't necessarily exist a continuous selection from Ψ, that is, a continuous function $f : X \to X$ such that $f(x) \in \Psi(x)$ for every $x \in X$. If such a function existed, then by applying Brouwer to f we would have a fixed point of Ψ (because if $\exists x^* \in X$ such that $x^* = f(x^*)$, then $x^* = f(x^*) \in \Psi(x^*)$).



$f(\bar{x}) \in \Psi(\bar{x})$?

? $\exists\, f: X \to X$ such that $f(x) \in \Psi(x)\ \forall x$
and f continuous

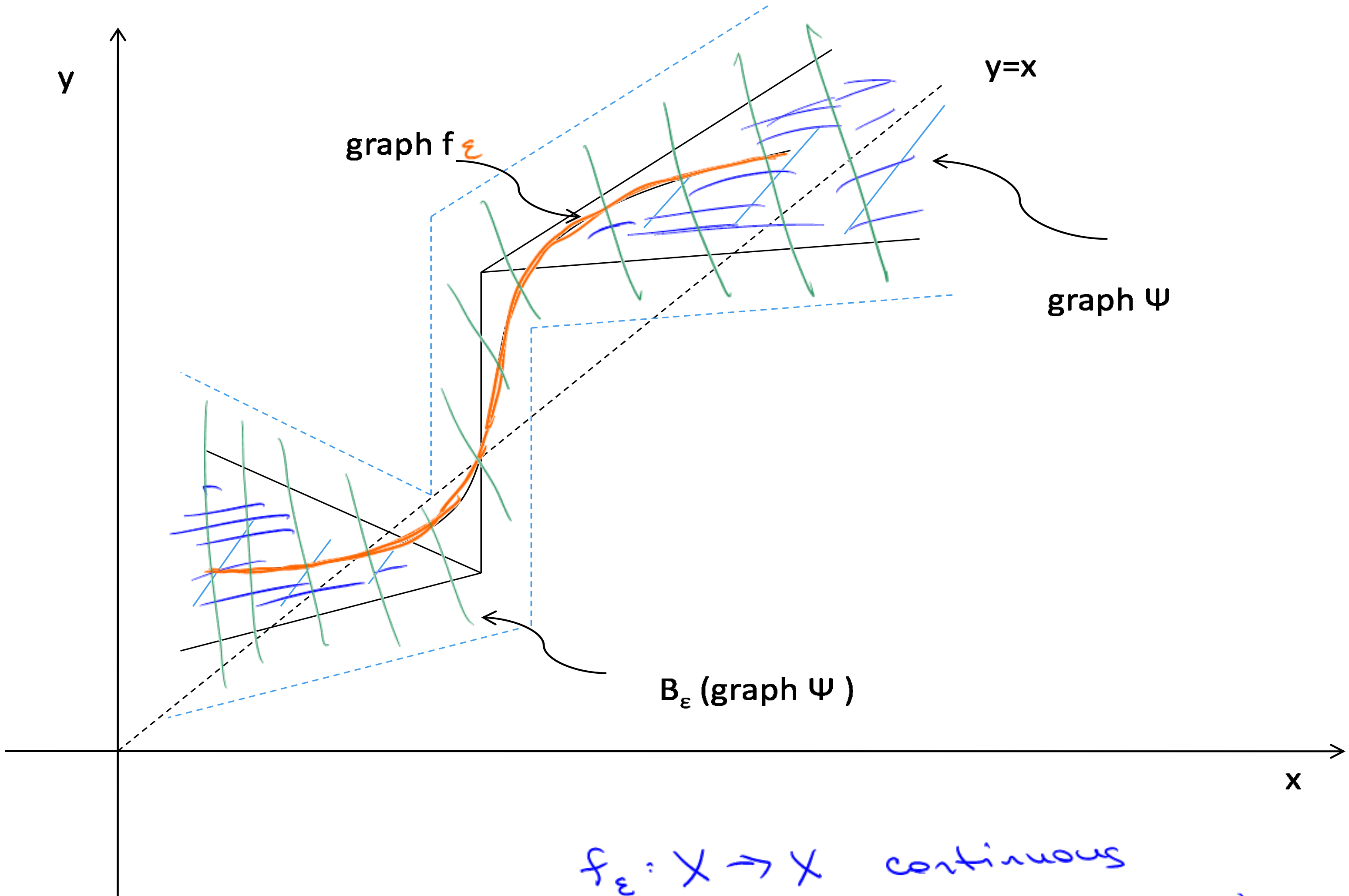
$\Psi(x)$ convex $\forall x \in X$, Ψ uhc

Instead, we look for a weaker type of approximation. Let $X \subset \mathbf{R}^n$ be a non-empty, compact, convex set, and let $\Psi : X \to 2^X$ be an uhc correspondence with non-empty, compact, convex values. For every $\varepsilon > 0$, define the ε ball about graph Ψ to be

$$B_\varepsilon(\text{ graph } \Psi) = \left\{ z \in X \times X : d(z, \text{ graph } \Psi) = \inf_{(x,y) \in \text{ graph } \Psi} d(z,(x,y)) < \varepsilon \right\}$$

Here d denotes the ordinary Euclidean distance. Since Ψ is uhc and convex-valued, for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : X \to X$ such that graph $f_\varepsilon \subseteq B_\varepsilon($ graph $\Psi)$.

$\varepsilon > 0$



$f_\varepsilon : X \to X$ continuous

graph $f_\varepsilon \subseteq B_\varepsilon(\text{graph } \Psi)$

$\varepsilon_n = \frac{1}{n} \ \forall n$ $\qquad f_n : X \to X \ \forall n$

Now by letting $\varepsilon \to 0$, this means that we can find a sequence of continuous functions $\{f_n\}$ such that graph $f_n \subseteq B_{\frac{1}{n}}($ graph $\Psi)$ for each n. By Brouwer's Fixed Point Theorem, each function f_n has a fixed point $\hat{x}_n \in X$, and

$$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \text{ graph } f_n \subseteq B_{\frac{1}{n}}(\text{ graph } \Psi) \text{ for each } n$$

So for each n there exists $(x_n, y_n) \in$ graph Ψ such that $d((\hat{x}_n, \hat{x}_n), (x_n, y_n)) < \frac{1}{n}$

$$d(\hat{x}_n, x_n) < \frac{1}{n} \text{ and } d(\hat{x}_n, y_n) < \frac{1}{n}$$

Since X is compact, $\{\hat{x}_n\}$ has a convergent subsequence $\{\hat{x}_{n_k}\}$, with $\hat{x}_{n_k} \to \hat{x} \in X$. Then $x_{n_k} \to \hat{x}$ and $y_{n_k} \to \hat{x}$. Since Ψ is uhc and closed-valued, it has closed graph, so $(\hat{x}, \hat{x}) \in$ graph Ψ. Thus $\hat{x} \in \Psi(\hat{x})$, that is, $\hat{x}$ is a fixed point of Ψ. $\square$

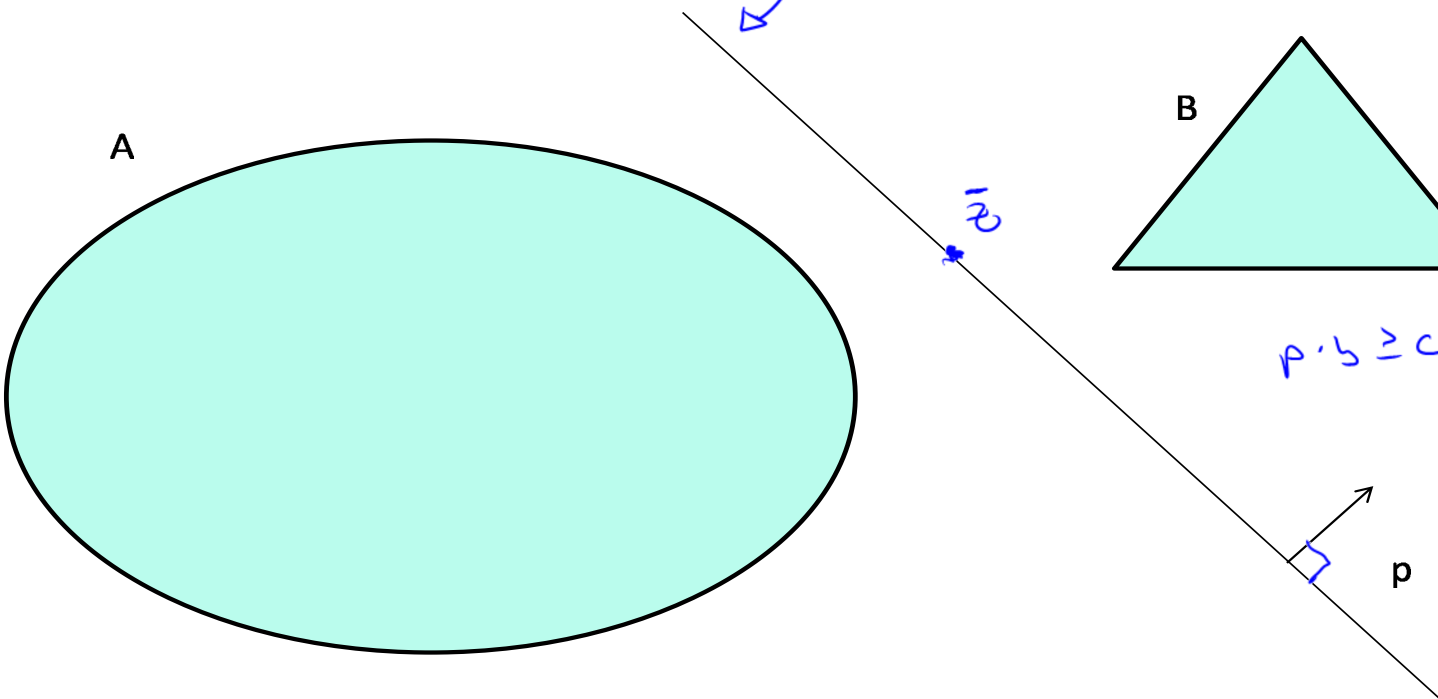
Separating Hyperplane Theorems

Theorem 4 (1.26, Separating Hyperplane Theorem)**.** *Let* $A, B \subseteq \mathbf{R}^n$ *be nonempty, disjoint convex sets. Then there exists a nonzero vector* $p \in \mathbf{R}^n$ *such that*

$$p \cdot a \leq p \cdot b \quad \forall a \in A, b \in B$$

hyperplane: $\{ z \in \mathbb{R}^n : v \cdot z = c \}$ for some $v \in \mathbb{R}^n$, $v \neq 0$,
and some $c \in \mathbb{R}$

$H = \{z \in \mathbb{R}^n ; p \cdot z = c\}$

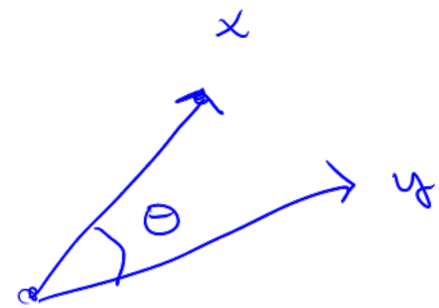


$p \cdot b \geq c \quad \forall b \in B$

$p \cdot a \leq c \quad \forall a \in A$

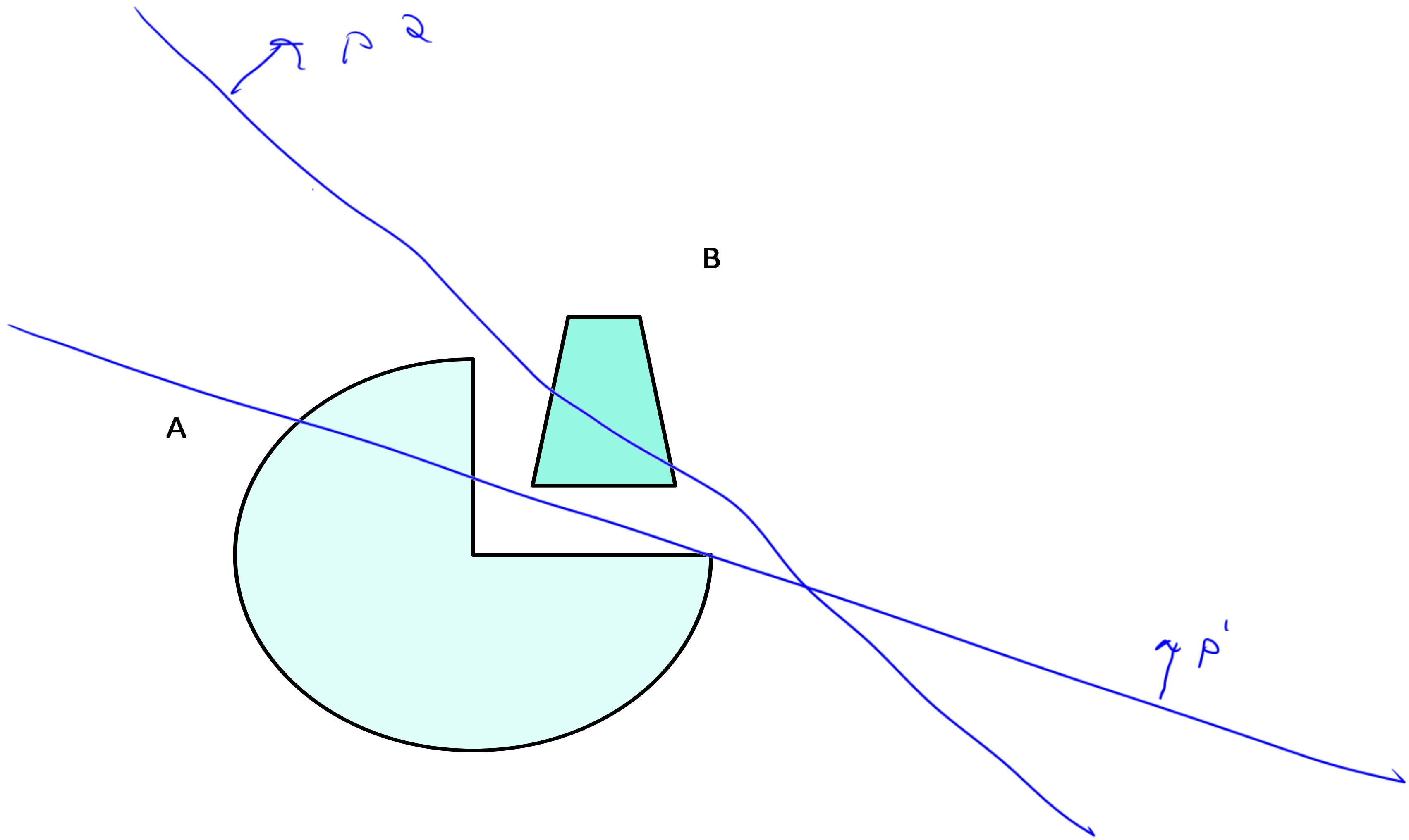
$c = p \cdot \bar{z}$

$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}$$

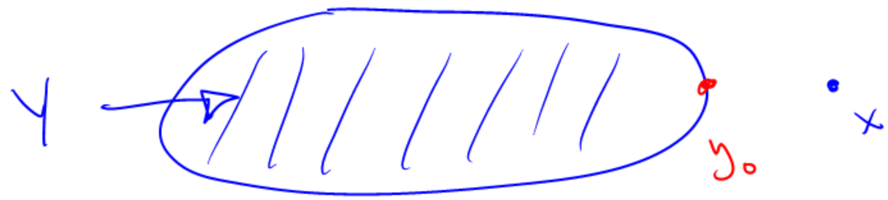


$$\cos\theta \geq 0 \iff \theta \leq 90^\circ$$

$$\cos\theta \leq 0 \iff \theta \geq 90^\circ$$



Convexity important: no hyperplane separates A and B



Separating a Point from a Set

Theorem 5. *Let* $Y \subseteq \mathbf{R}^n$ *be a nonempty convex set and* $x \notin Y$*. Then there exists a nonzero vector* $p \in \mathbf{R}^n$ *such that*

$$p \cdot x \leq p \cdot y \quad \forall y \in Y$$

Proof. We sketch the proof in the special case that Y is compact. We will see that in this case we actually get a stronger conclusion:

$$\exists p \in \mathbf{R}^n,\ p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

Choose $y_0 \in Y$ such that $\|y_0 - x\| = \inf\{\|y - x\| : y \in Y\}$; such a point exists because Y is compact, so the distance function $g(y) = \|y - x\|$ assumes its minimum on Y. Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$. Let $p = y_0 - x$. The set

g is continuous

$$H = \{z \in \mathbf{R}^n : p \cdot z = p \cdot y_0\}$$

is the hyperplane perpendicular to p through y_0. See Figure 12. Then

$$\begin{aligned} p \cdot y_0 &= (y_0 - x) \cdot y_0 \\ &= (y_0 - x) \cdot (y_0 - x + x) \\ &= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x \\ &= \|y_0 - x\|^2 + p \cdot x \\ &> p \cdot x \end{aligned}$$

We claim that

$$\forall \quad y \in Y \Rightarrow p \cdot y \geq p \cdot y_0 > p \cdot x$$

If not, suppose there exists $y \in Y$ such that $p \cdot y < p \cdot y_0$. Given $\alpha \in (0, 1)$, let

$$w_\alpha = \alpha y + (1 - \alpha) y_0$$

Since Y is convex, $w_\alpha \in Y$. Then for α sufficiently close to zero,

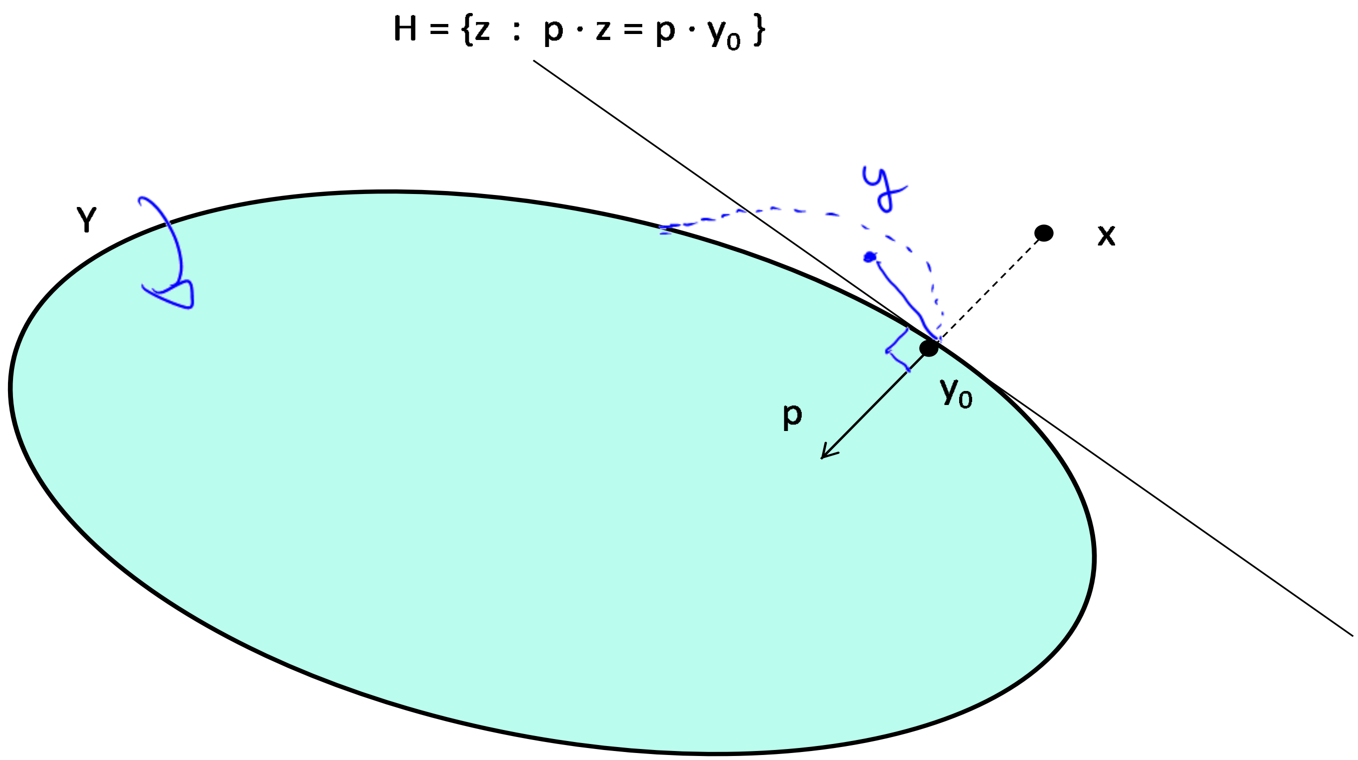
$$
\begin{aligned}
\|x - w_\alpha\|^2 &= \|x - \alpha y - (1-\alpha)y_0\|^2 && \text{defn of } w_\alpha \\
&= |x - y_0 + \alpha(y_0 - y)|^2 && \text{algebra} \\
&= |-p + \alpha(y_0 - y)|^2 && \text{defn of } p \\
&= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2 && \text{more algebra} \\
&= |p|^2 + \alpha\left(\underbrace{-2p \cdot (y_0 - y)}_{\text{negative}} + \underbrace{\alpha |y_0 - y|^2}_{\to 0 \text{ as } \alpha \to 0}\right) && \text{''} \\
&< |p|^2 \quad \text{for } \alpha \text{ close to } 0, \text{ as } p \cdot y_0 > p \cdot y \\
&= \|y_0 - x\|^2
\end{aligned}
$$

Thus for α sufficiently close to zero,

$$\|w_\alpha - x\| < \|y_0 - x\|$$

which implies y_0 is not the closest point in Y to x, contradiction.

$\square$


H = {z : p · z = p · y0 }
Y
y
x
y0
p

The general version of the Separating Hyperplane Theorem can be derived from this special case by noting that if $A \cap B = \emptyset$, then $0 \notin A - B = \{a - b : a \in A, b \in B\}$.

$A - B$ not closed if

A, B closed

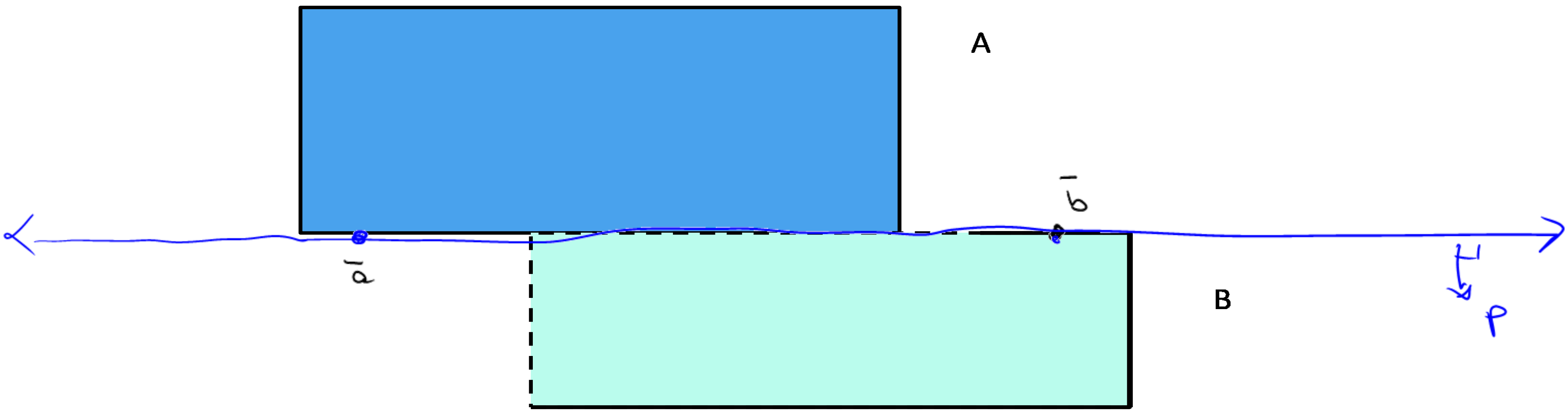
not necessarily bdd

Strict Separation

For the special case of Y compact and $X = \{x\}$, we actually could *strictly separate* Y and X:

$$x \notin Y \Rightarrow \quad \exists p \in \mathbf{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

When can we do this in general? Will require additional assumptions...



A, B nonempty, disjoint, convex $\Rightarrow$

$\exists p \in \mathbb{R}^n$, $p \neq 0$ s.t. $p \cdot a \leq p \cdot b$ $\forall a \in A$ $\forall b \in B$

But

$p \cdot \bar{a} = p \cdot \bar{b}$ for some $\bar{a} \in A$ and $\bar{b} \in B$

(for any such p)

Strict Separation

Theorem 6. (Strict Separating Hyperplane Theorem) *Let $A, B \subseteq \mathbf{R}^n$ be nonempty, disjoint, convex sets with A closed and B compact. Then there exists a nonzero vector $p \in \mathbf{R}^n$ such that*

$$p \cdot a < p \cdot b \quad \forall a \in A, b \in B$$