

Econ 204 2023

Lecture 3

Outline

1. Metric Spaces and Normed Spaces
2. Convergence of Sequences in Metric Spaces
3. Sequences in \mathbf{R} and \mathbf{R}^n

Metric Spaces and Metrics

Generalize distance and length notions in \mathbf{R}^n

Definition 1. A metric space is a pair (X, d) , where X is a set and $d : X \times X \rightarrow \mathbf{R}_+$ a function satisfying

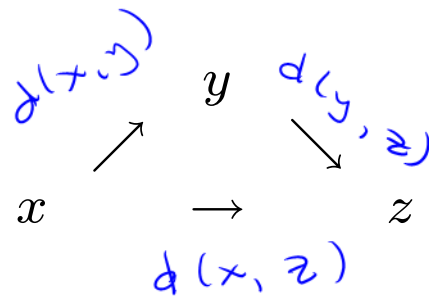
$$\mathbf{R}_+ = \{r \in \mathbf{R} : r \geq 0\}$$

1. $d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$

2. $d(x, y) = d(y, x) \quad \forall x, y \in X$

3. triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$$



A function $d : X \times X \rightarrow \mathbf{R}_+$ satisfying 1-3 above is called a metric on X .

A metric gives a notion of distance between elements of X .

$$f: V \rightarrow \mathbb{R}_+ \quad \|x\|$$

$$\| \cdot \| = f$$

Normed Spaces and Norms

Definition 2. Let V be a vector space over \mathbb{R} . A norm on V is a function $\| \cdot \| : V \rightarrow \mathbb{R}_+$ satisfying

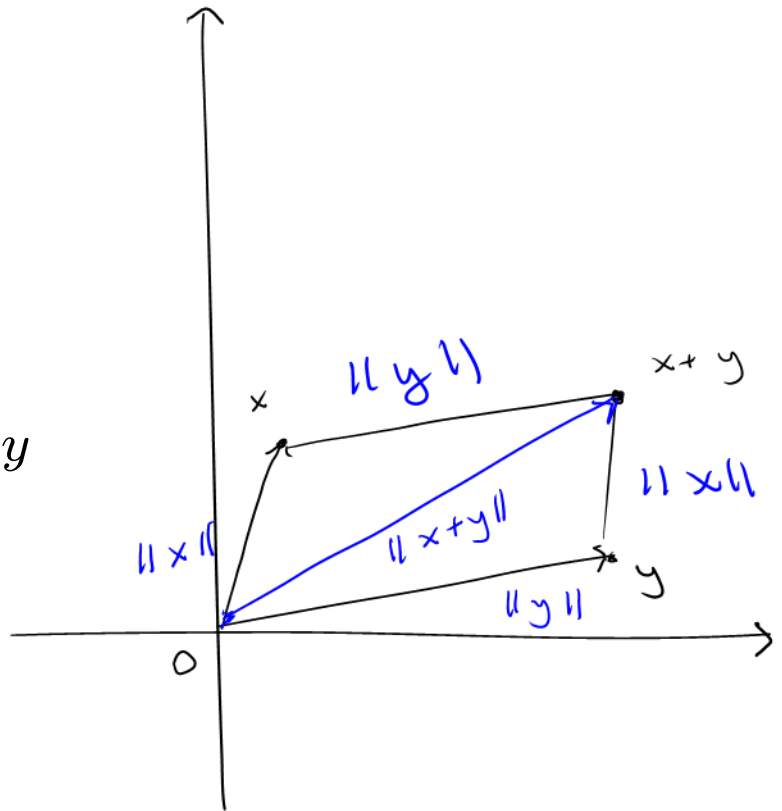
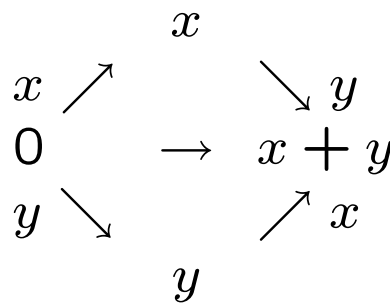
1. $\|x\| \geq 0 \quad \forall x \in V$

2. $\|x\| = 0 \Leftrightarrow x = \underline{0} \quad \forall x \in V$

vector additive identity

3. triangle inequality:

$$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$$



$$4. \|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in \mathbf{R}, x \in V$$

A normed vector space is a vector space over \mathbf{R} equipped with a norm.

($\forall, \|\cdot\|$)

A norm gives a notion of length of a vector in V .

$$x \in \mathbb{R}^n \quad x = (x_1, \dots, x_n) \quad \|x\| = \left[\sum_{i=1}^n x_i^2 \right]^{1/2}$$

Normed Spaces and Norms

Example: In \mathbb{R}^n , standard notion of distance between two vectors x and y measures length of difference $x - y$, i.e.,
$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

Theorem 1. Let $(V, \|\cdot\|)$ be a normed vector space. Let $d : V \times V \rightarrow \mathbb{R}_+$ be defined by

$$d(v, w) = \|v - w\|$$

Then (V, d) is a metric space.

Proof. We must verify that d satisfies all the properties of a metric.

1. Let $v, w \in V$. Then by definition, $d(v, w) = \|v - w\| \geq 0$ (why?), and

$$\begin{aligned}
 d(v, w) = 0 &\Leftrightarrow \|v - w\| = 0 \\
 &\Leftrightarrow v - w = 0 \\
 &\Leftrightarrow (v + (-w)) + w = w \\
 &\Leftrightarrow v + ((-w) + w) = w \\
 &\Leftrightarrow v + 0 = w \\
 &\Leftrightarrow v = w
 \end{aligned}$$

vector additive identity

$$\begin{aligned}
 0 \cdot x &= 0 \\
 -1 \cdot x &= -x
 \end{aligned}$$

symmetry: 2. First, note that for any $x \in V$, $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$, so $0 \cdot x = 0$. Then $0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x =$

vector additive identity

x

$0 = x + (-1) \cdot x$, so we have $(-1) \cdot x = (-x)$. Then let $v, w \in V$.

$$\begin{aligned}d(v, w) &= \|v - w\| \\&= |-1| \|v - w\| \\&= \|(-1)(v + (-w))\| \\&= \|(-1)v + (-1)(-w)\| \\&= \|-v + w\| \\&= \|w + (-v)\| \\&= \|w - v\| \\&= d(w, v)\end{aligned}$$

triangle inequality

3. Let $u, w, v \in V$.

$$\begin{aligned}d(u, w) &= \|u - w\| \\&= \|u + (-v + v) - w\| \\&= \|(u - v) + (v - w)\| \\&\leq \|u - v\| + \|v - w\| \\&= d(u, v) + d(v, w)\end{aligned}$$

Thus d is a metric on V .



$$x = (x_1, \dots, x_n) \quad x_i \in \mathbb{R}$$

x_i

Normed Spaces and Norms

Examples

- \mathbb{E}^n : n -dimensional Euclidean space. *\mathbb{R}^n with standard norm*

$$V = \mathbf{R}^n, \quad \|x\|_2 = |x| = \sqrt{\sum_{i=1}^n (x_i)^2}$$

- $V = \mathbf{R}^n, \quad \|x\|_1 = \sum_{i=1}^n |x_i|$ (the “taxi cab” norm or L^1 norm)
- $V = \mathbf{R}^n, \quad \|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$ (the maximum norm, or sup norm, or L^∞ norm)

Recall: $C([0,1])$ - continuous functions $f: [0,1] \rightarrow \mathbb{R}$

$\checkmark =$

- $C([0,1]), \|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$

$\checkmark =$

- $C([0,1]), \|f\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$

$\checkmark =$

- $C([0,1]), \|f\|_1 = \int_0^1 |f(t)| dt$

$$v, w \in \mathbb{R}^n \quad \langle v, w \rangle = v \cdot w = \sum_{i=1}^n v_i w_i$$

Normed Spaces and Norms

Theorem 2 (Cauchy-Schwarz Inequality).

If $v, w \in \mathbb{R}^n$, then

$$\left(\sum_{i=1}^n v_i w_i \right)^2 \leq \left(\sum_{i=1}^n v_i^2 \right) \left(\sum_{i=1}^n w_i^2 \right)$$

$$|\langle v, w \rangle|^2 = |v \cdot w|^2 \leq |v|^2 |w|^2 = \|v\|^2 \|w\|^2$$

$$|\langle v, w \rangle| = |v \cdot w| \leq |v| |w| = \|v\| \|w\|$$

- learn some proof
- triangle inequality of $\|\cdot\|_2$ in \mathbb{R}^n follows from C-S inequality (nice exercise)

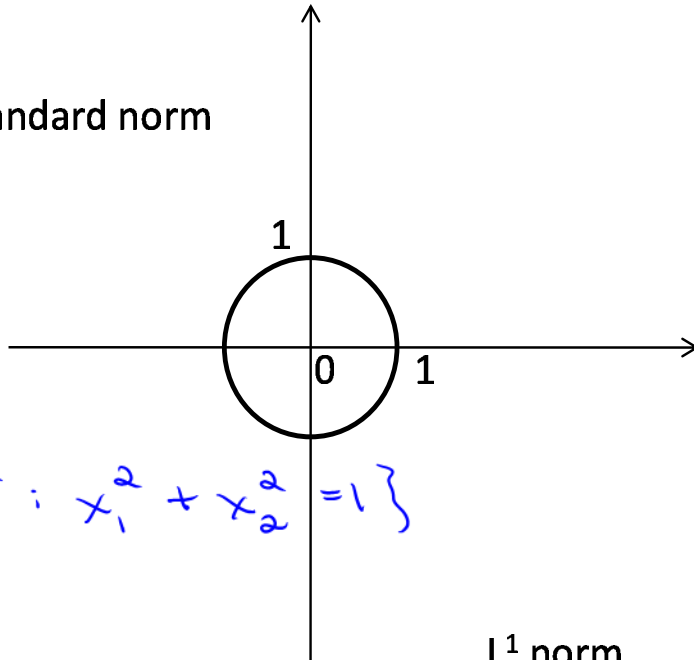
Equivalent Norms

A given vector space may have many different norms: if $\|\cdot\|$ is a norm on a vector space V , so are $2\|\cdot\|$ and $3\|\cdot\|$ and $k\|\cdot\|$ for any $k > 0$.

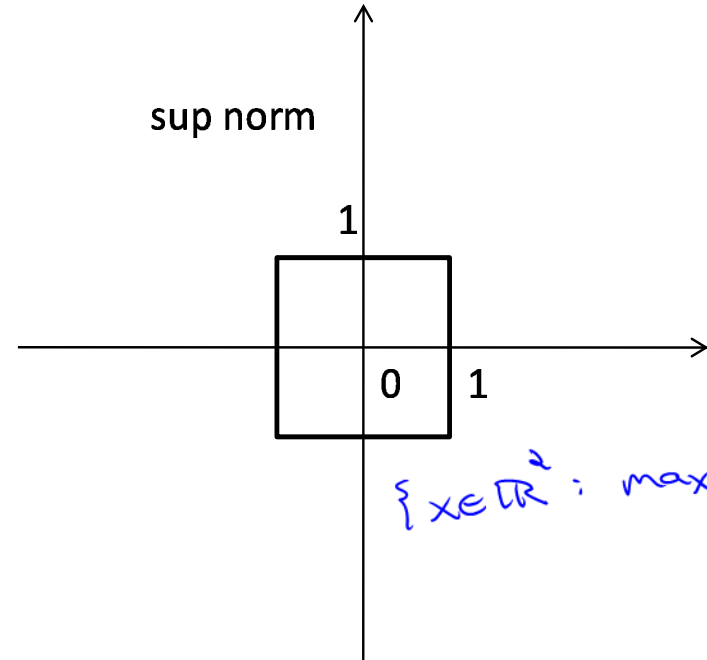
Less trivially, \mathbf{R}^n supports many different norms as in the examples above. Different norms on a given vector space yield different geometric properties.

$\{x \in \mathbb{R}^2 : \|x\|=1\}$ for different norms :

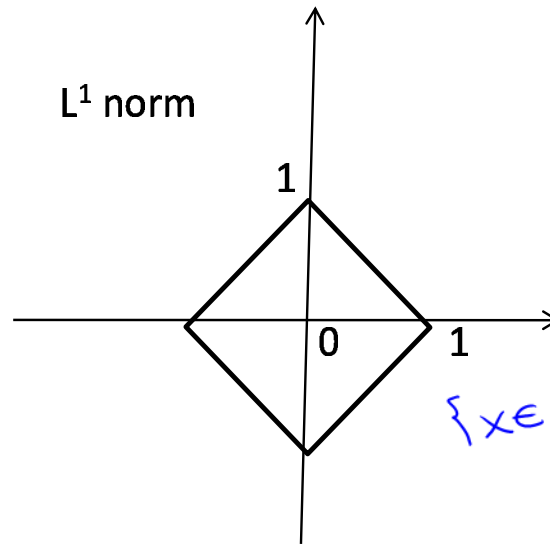
standard norm



sup norm



L^1 norm



unit balls around 0 in different norms

Equivalent Norms

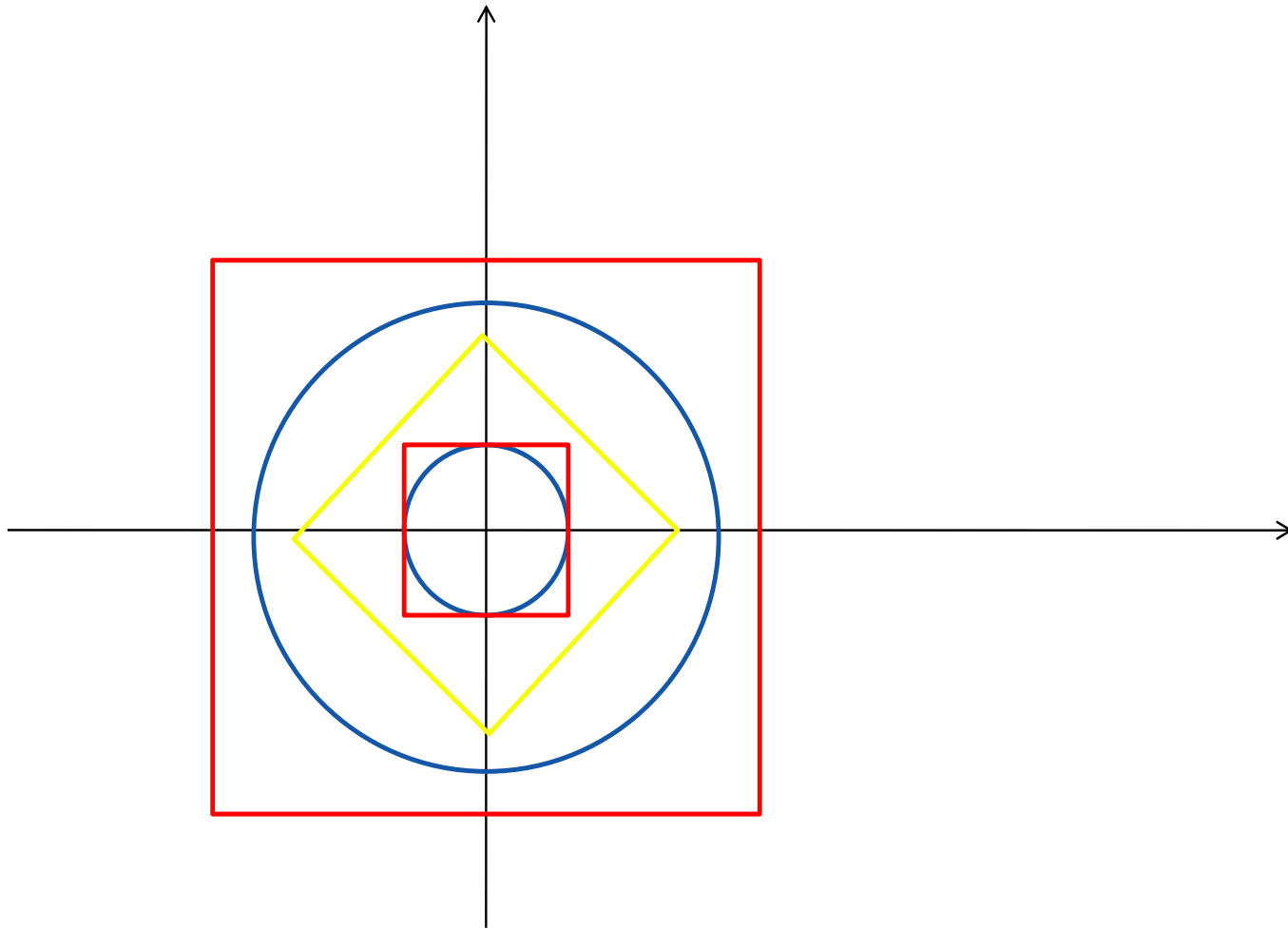
Definition 3. Two norms $\| \cdot \|$ and $\| \cdot \|^{*}$ on the same vector space V are said to be Lipschitz-equivalent (or equivalent) if $\exists m, M > 0$ s.t. $\forall x \in V$,

$$m\|x\| \leq \|x\|^{*} \leq M\|x\|$$

Equivalently, $\exists m, M > 0$ s.t. $\forall x \in V, x \neq 0,$

$$0 < m \leq \frac{\|x\|^{*}}{\|x\|} \leq M < +\infty$$

this is an equivalence relation (nice exercise)



norms on \mathbf{R}^n are equivalent

Equivalent Norms

In \mathbf{R}^n (or any finite-dimensional normed vector space), all norms are equivalent. Roughly, up to a difference in scaling, for topological purposes there is a unique norm in \mathbf{R}^n .

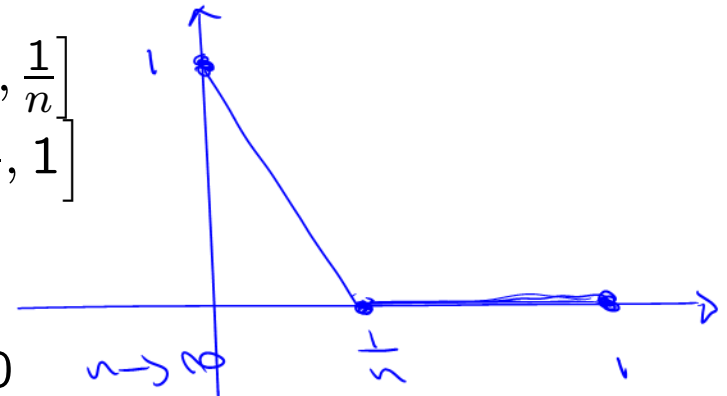
Theorem 3. *All norms on \mathbf{R}^n are equivalent.*

Infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0, 1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt & \text{if } t \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } t \in \left(\frac{1}{n}, 1\right] \end{cases}$$

Then

$$\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{\frac{1}{2n}}{1} = \frac{1}{2n} \rightarrow 0 \quad n \rightarrow \infty$$



$$\|f_n\|_1 = \int_0^1 |f_n(t)| dt = \frac{1}{2n}$$

$$\|f_n\|_\infty = \sup \{ |f_n(t)| : t \in [0, 1] \} = 1 \quad \forall n$$

Metrics and Sets

Definition 4. In a metric space (X, d) , a subset $S \subseteq X$ is bounded if $\exists x \in X, \beta \in \mathbf{R}$ such that $\forall s \in S, d(s, x) \leq \beta$.

In a metric space (X, d) , define for $\varepsilon > 0$

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X : d(y, x) < \varepsilon\} \\ &= \text{"open" ball with center } x \text{ and radius } \varepsilon \end{aligned}$$

$$\begin{aligned} B_\varepsilon[x] &= \{y \in X : d(y, x) \leq \varepsilon\} \\ &= \text{"closed" ball with center } x \text{ and radius } \varepsilon \end{aligned}$$

$$C = \{x \in \mathbb{R} : 0 < x < 1\}$$

$$\exists y \in C \text{ s.t. } y \geq x \quad \forall x \in C$$

Metrics and Sets



We can use the metric d to define a generalization of “radius”. In a metric space (X, d) , define the *diameter* of a subset $S \subseteq X$ by

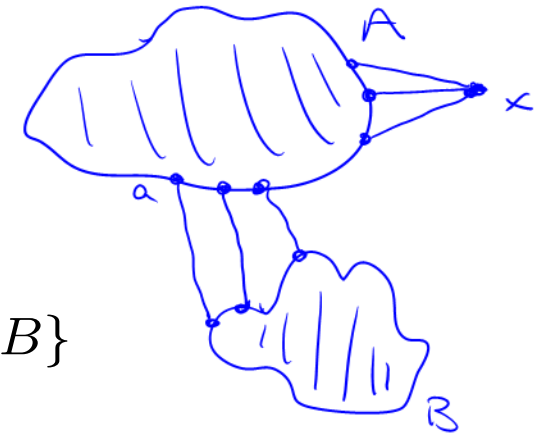
$$\text{diam}(S) = \sup\{d(s, s') : s, s' \in S\}$$

Similarly, we can define the distance from a point to a set, and distance between sets, as follows:

$$d(A, x) = \inf_{a \in A} d(a, x)$$

$$d(A, B) = \inf_{a \in A} d(B, a)$$

$$= \inf\{d(a, b) : a \in A, b \in B\}$$



2^x
 $x \cdot 2^x$

But $d(A, B)$ is **not** a metric.

$\forall a \exists b \text{ s.t. } \dots$

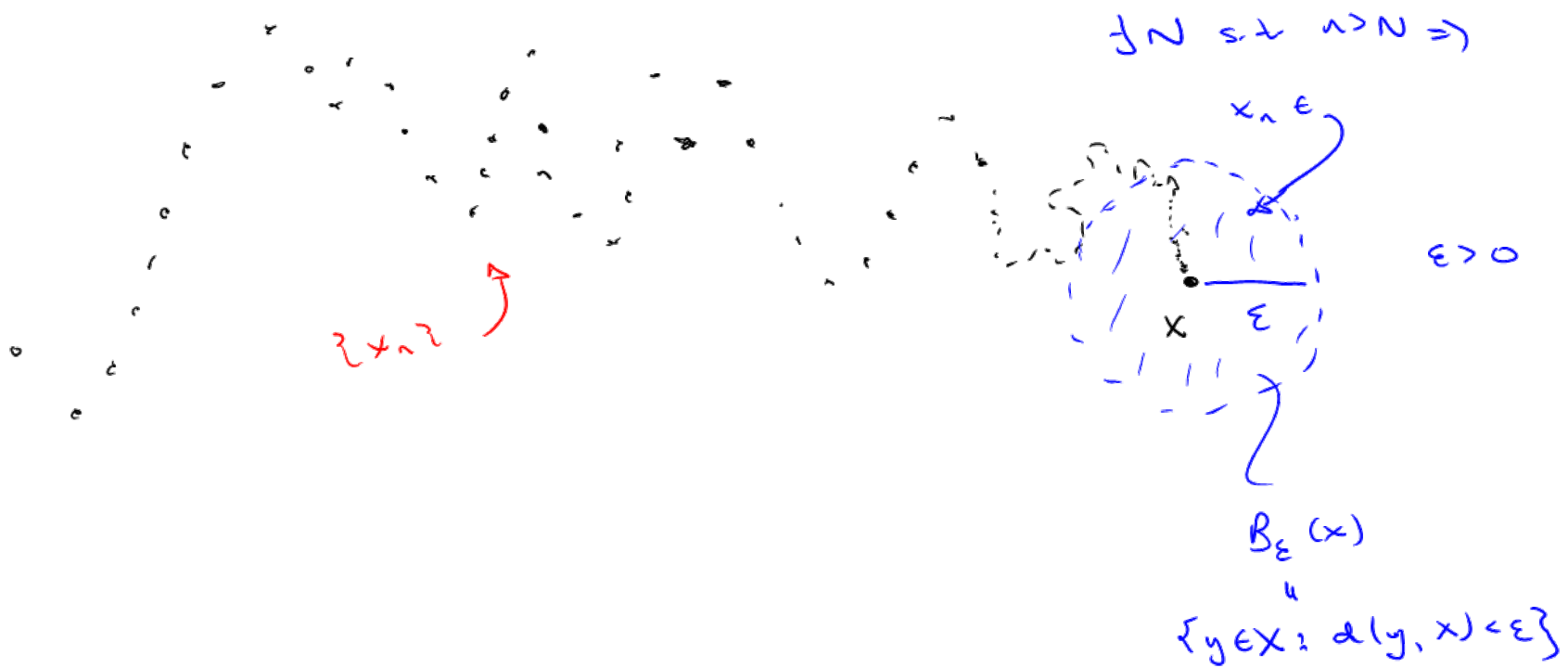
Convergence of Sequences

Definition 5. Let (X, d) be a metric space. A sequence $\{x_n\}$ converges to x (written $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$) if

$$\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbf{N} \text{ s.t. } n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$$

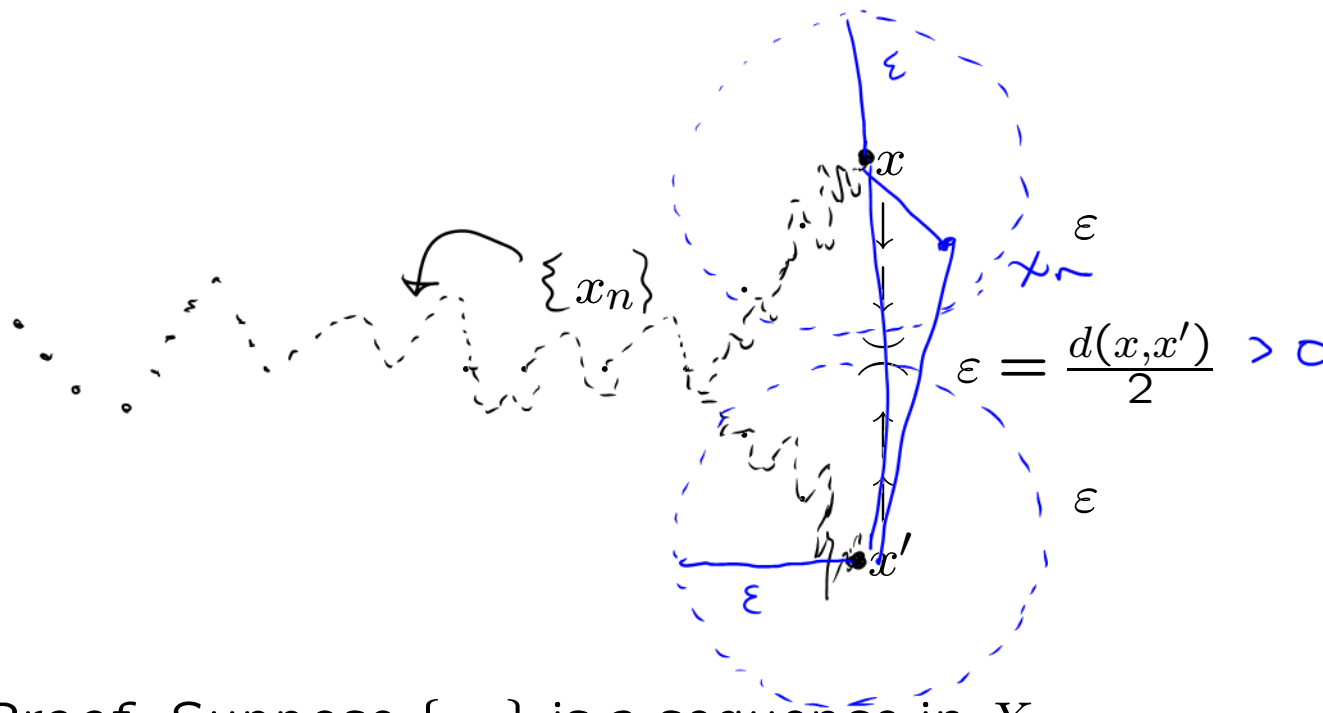
Notice that this is exactly the same as the definition of convergence of a sequence of real numbers, except we replace the standard measure of distance $|\cdot|$ in \mathbf{R} by the general metric d .

$\forall \varepsilon > 0 \exists N \text{ s.t. } \forall n > N \ d(x_n, x) < \varepsilon$



Uniqueness of Limits

Theorem 4 (Uniqueness of Limits). *In a metric space (X, d) , if $x_n \rightarrow x$ and $x_n \rightarrow x'$, then $x = x'$.*



Proof. Suppose $\{x_n\}$ is a sequence in X , $x_n \rightarrow x$, $x_n \rightarrow x'$, $x \neq x'$.

Since $x \neq x'$, $d(x, x') > 0$. Let

$$\varepsilon = \frac{d(x, x')}{2} > 0$$

Then there exist $N(\varepsilon)$ and $N'(\varepsilon)$ such that

$$n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon \quad (x_n \rightarrow x)$$

$$n > N'(\varepsilon) \Rightarrow d(x_n, x') < \varepsilon \quad (x_n \rightarrow x')$$

Choose

$$n > \max\{N(\varepsilon), N'(\varepsilon)\}$$

Fix $n > \max \{N, N'\}$

Then

$$d(x, x') \leq d(x, x_n) + d(x_n, x')$$

$$< \varepsilon + \varepsilon$$

$$= 2\varepsilon$$

$$= d(x, x')$$

$$\Rightarrow d(x, x') < d(x, x')$$

a contradiction.



$$\{y \in X : d(y, c) < \varepsilon\}$$

Cluster Points

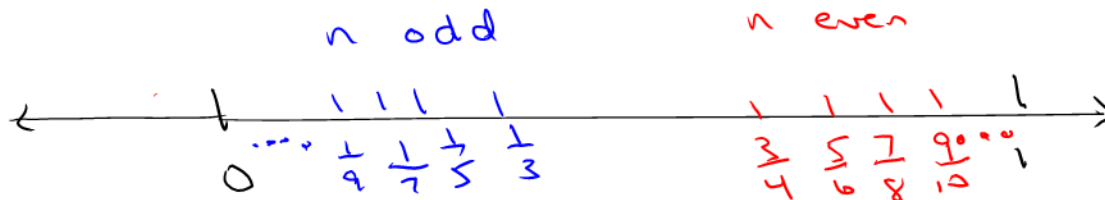
Definition 6. An element c is a cluster point of a sequence $\{x_n\} \subseteq X$ in a metric space (X, d) if $\forall \varepsilon > 0$, $\{n : x_n \in B_\varepsilon(c)\}$ is an infinite set. Equivalently,

$$\forall \varepsilon > 0, N \in \mathbb{N} \exists n > N \text{ s.t. } x_n \in B_\varepsilon(c)$$

Example:

$$x_n = \begin{cases} 1 - \frac{1}{n} & \text{if } n \text{ even} \\ \frac{1}{n} & \text{if } n \text{ odd} \end{cases}$$

For n large and odd, x_n is close to zero; for n large and even, x_n is close to one. The sequence does not converge; the set of cluster points is $\{0, 1\}$.



Subsequences

If $\{x_n\}$ is a sequence and $n_1 < n_2 < n_3 < \dots$ then $\{x_{n_k}\}$ is called a *subsequence*. *of $\{x_n\}$*

= infinitely many

Note that a subsequence is formed by taking some of the elements of the parent sequence, *in the same order*.

Example: $x_n = \frac{1}{n}$, so $\{x_n\} = (1, \frac{1}{2}, \frac{1}{3}, \dots)$. If $n_k = 2k$, then $\{x_{n_k}\} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots)$.

Cluster Points and Subsequences

Theorem 5 (2.4 in De La Fuente, plus ...). *Let (X, d) be a metric space, $c \in X$, and $\{x_n\}$ a sequence in X . Then c is a cluster point of $\{x_n\}$ if and only if there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \rightarrow \infty} x_{n_k} = c$.*

\Rightarrow : *Proof.* Suppose c is a cluster point of $\{x_n\}$. We inductively construct a subsequence that converges to c . For $k = 1$, $\{n : x_n \in B_1(c)\}$ is infinite, so nonempty; let

$$n_1 = \min\{n : x_n \in B_1(c)\}$$

Now, suppose we have chosen $n_1 < n_2 < \dots < n_k$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \dots, k$$

$\{n : x_n \in B_{\frac{1}{k+1}}(c)\}$ is infinite, so it contains at least one element bigger than n_k , so let

$$n_{k+1} = \min \left\{ n : n > n_k, x_n \in B_{\frac{1}{k+1}}(c) \right\}$$

Thus, we have chosen $n_1 < n_2 < \dots < n_k < n_{k+1}$ such that

$$x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \dots, k, k+1$$

Thus, by induction, we obtain a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \in B_{\frac{1}{k}}(c) \quad \forall k$$

Given any $\varepsilon > 0$, by the Archimedean property, there exists $N(\varepsilon) > 1/\varepsilon$.

$$\begin{aligned} k > N(\varepsilon) &\Rightarrow x_{n_k} \in B_{\frac{1}{k}}(c) \\ &\Rightarrow x_{n_k} \in B_{\varepsilon}(c) \end{aligned}$$

$$k > N \Rightarrow \frac{1}{k} < \varepsilon$$

so

$$x_{n_k} \rightarrow c \text{ as } k \rightarrow \infty$$

\Leftarrow : Conversely, suppose that there is a subsequence $\{x_{n_k}\}$ converging to c . Given any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$k > K \Rightarrow d(x_{n_k}, c) < \varepsilon \Rightarrow x_{n_k} \in B_\varepsilon(c)$$

Therefore,

$$\{n : x_n \in B_\varepsilon(c)\} \supseteq \{n_{K+1}, n_{K+2}, n_{K+3}, \dots\}$$

Since $n_{K+1} < n_{K+2} < n_{K+3} < \dots$, this set is infinite, so c is a cluster point of $\{x_n\}$. \square

Sequences in \mathbf{R} and \mathbf{R}^m

Definition 7. A sequence of real numbers $\{x_n\}$ is increasing (decreasing) if $x_{n+1} \geq x_n$ ($x_{n+1} \leq x_n$) for all n .

Definition 8. If $\{x_n\}$ is a sequence of real numbers, $\{x_n\}$ tends to infinity (written $x_n \rightarrow \infty$ or $\lim x_n = \infty$) if

$$\forall K \in \mathbf{R} \exists N(K) \text{ s.t. } n > N(K) \Rightarrow x_n > K$$

Similarly define $x_n \rightarrow -\infty$ or $\lim x_n = -\infty$.

Increasing and Decreasing Sequences

Theorem 6 (Theorem 3.1'). *Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. Then*

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbf{N}\}$$

$$\left(\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbf{N}\} \right)$$

In particular, the limit exists.

work through proof in dlf - think about unbounded case

Lim Sups and Lim Infs

Consider a sequence $\{x_n\}$ of real numbers. Let

$$\alpha_n = \sup\{x_k : k \geq n\} \quad \text{fin}$$

$$= \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$$

$$\beta_n = \inf\{x_k : k \geq n\} \quad \text{fin}$$

$$= \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$$

Either $\alpha_n = +\infty$ for all n , or $\alpha_n \in \mathbf{R}$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots$.

Either $\beta_n = -\infty$ for all n , or $\beta_n \in \mathbf{R}$ and $\beta_1 \leq \beta_2 \leq \beta_3 \leq \dots$.

Lim Sups and Lim Infs

Definition 9.

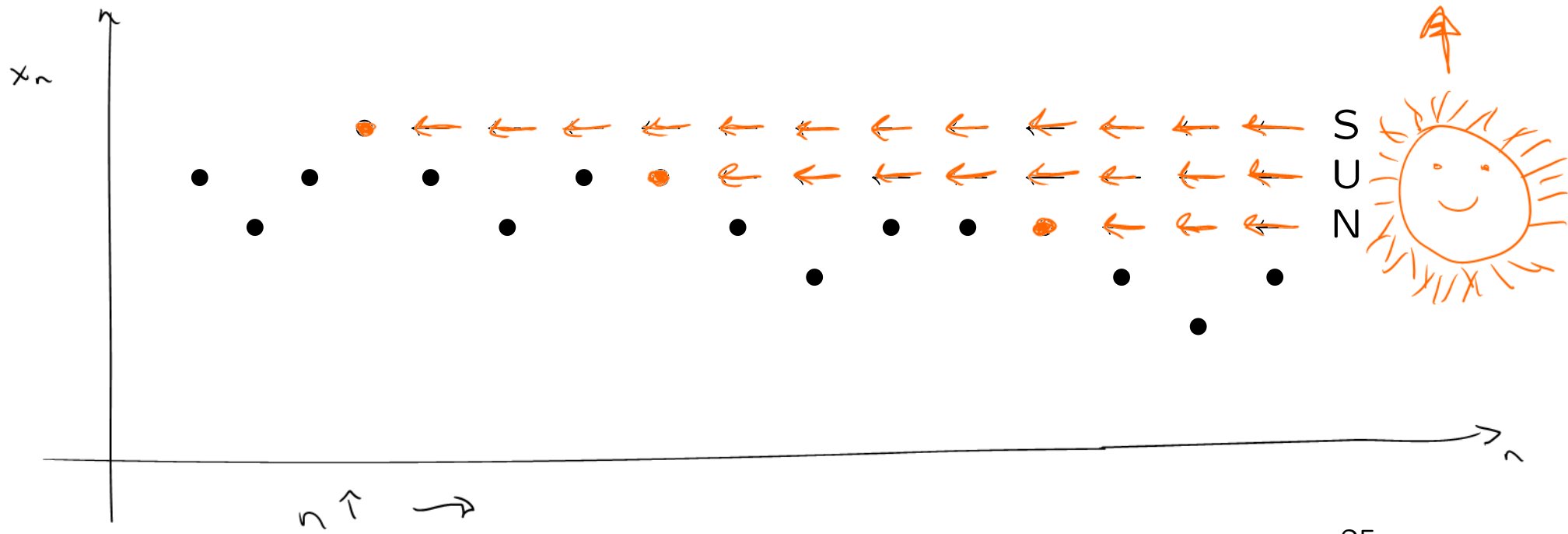
$$\begin{aligned}\limsup_{n \rightarrow \infty} x_n &= \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases} \\ \liminf_{n \rightarrow \infty} x_n &= \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}\end{aligned}$$

Theorem 7. *Let $\{x_n\}$ be a sequence of real numbers. Then*

$$\begin{aligned}\lim_{n \rightarrow \infty} x_n &= \gamma \in \mathbf{R} \cup \{-\infty, \infty\} \\ \Leftrightarrow \limsup_{n \rightarrow \infty} x_n &= \liminf_{n \rightarrow \infty} x_n = \gamma\end{aligned}$$

Increasing and Decreasing Subsequences

Theorem 8 (Theorem 3.2, Rising Sun Lemma). *Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.*



Proof. Let

$$S = \{s \in \mathbf{N} : x_s > x_n \quad \forall n > s\}$$

Either S is infinite, or S is finite. (or empty)

If S is infinite, let

$$\begin{aligned} n_1 &= \min S \\ n_2 &= \min (S \setminus \{n_1\}) \\ n_3 &= \min (S \setminus \{n_1, n_2\}) \\ &\vdots \\ n_{k+1} &= \min (S \setminus \{n_1, n_2, \dots, n_k\}) \end{aligned}$$

Then $n_1 < n_2 < n_3 < \dots$.

$$\begin{array}{ll} x_{n_1} > x_{n_2} & \text{since } n_1 \in S \text{ and } n_2 > n_1 \\ x_{n_2} > x_{n_3} & \text{since } n_2 \in S \text{ and } n_3 > n_2 \\ & \vdots \\ x_{n_k} > x_{n_{k+1}} & \text{since } n_k \in S \text{ and } n_{k+1} > n_k \\ & \vdots \end{array}$$

so $\{x_{n_k}\}$ is a strictly decreasing subsequence of $\{x_n\}$.

If S is finite and nonempty, let $n_1 = (\max S) + 1$; if $S = \emptyset$, let $n_1 = 1$. Then

$$\begin{array}{ll} n_1 \notin S & \text{so } \exists n_2 > n_1 \text{ s.t. } x_{n_2} \geq x_{n_1} \\ n_2 \notin S & \text{so } \exists n_3 > n_2 \text{ s.t. } x_{n_3} \geq x_{n_2} \\ & \vdots \\ n_k \notin S & \text{so } \exists n_{k+1} > n_k \text{ s.t. } x_{n_{k+1}} \geq x_{n_k} \\ & \vdots \end{array}$$

so $\{x_{n_k}\}$ is a (weakly) increasing subsequence of $\{x_n\}$.



Bolzano-Weierstrass Theorem

Theorem 9 (Thm. 3.3, Bolzano-Weierstrass). *Every bounded sequence of real numbers contains a convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence of real numbers. By the Rising Sun Lemma, find an increasing or decreasing subsequence $\{x_{n_k}\}$. If $\{x_{n_k}\}$ is increasing, then by Theorem 3.1',

$$\lim x_{n_k} = \sup\{x_{n_k} : k \in \mathbf{N}\} \leq \sup\{x_n : n \in \mathbf{N}\} < \infty$$

since the sequence is bounded; since the limit is finite, the subsequence converges. Similarly, if the subsequence is decreasing, it converges. □