Announcements

- PS 1 due tomorrow
- read dLF on limits of functions

Econ 204 2023

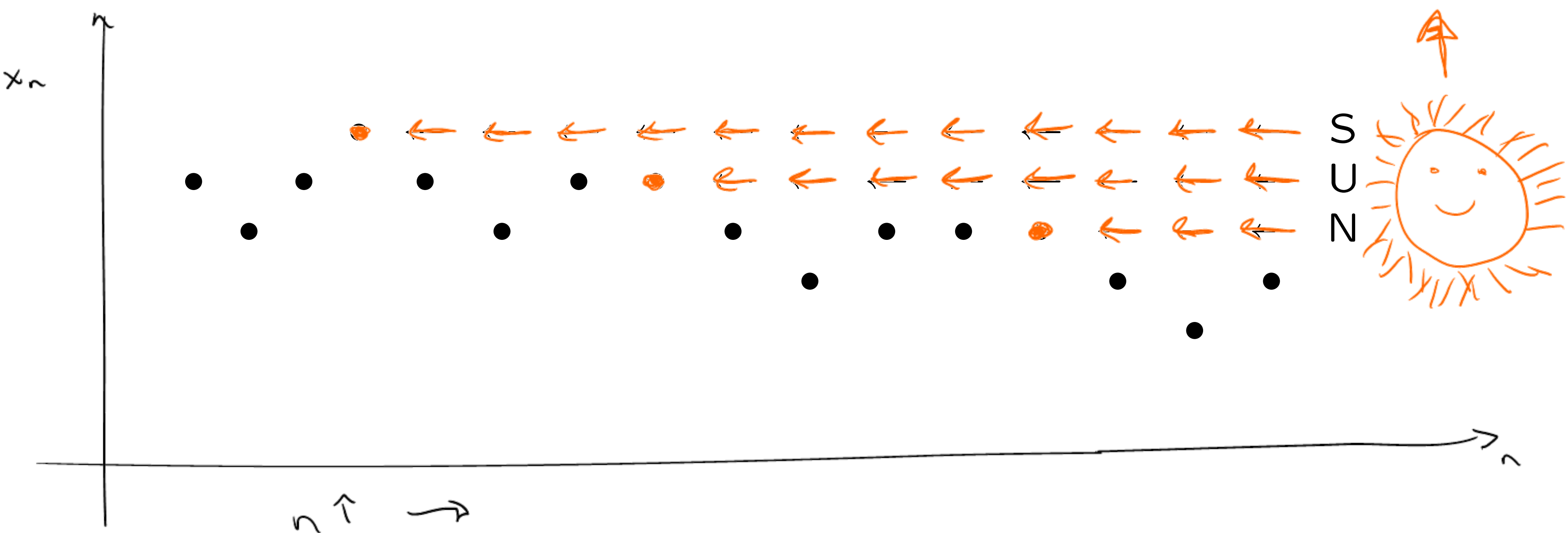
Lecture 4

Outline

0. Sequences in $\mathbb{R}$ (cont)
1. Open and Closed Sets
2. Continuity in Metric Spaces

Increasing and Decreasing Subsequences

Theorem 8 (Theorem 3.2, Rising Sun Lemma)**.** *Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.*



Proof. Let

$$S = \{s \in \mathbf{N} : x_s > x_n \quad \forall n > s\}$$

Either S is infinite, or S is finite. (or empty)

If S is infinite, let

$$\begin{aligned} n_1 &= \min S \\ n_2 &= \min (S \setminus \{n_1\}) \\ n_3 &= \min (S \setminus \{n_1, n_2\}) \\ &\vdots \\ n_{k+1} &= \min (S \setminus \{n_1, n_2, \ldots, n_k\}) \end{aligned}$$

Then $n_1 < n_2 < n_3 < \cdots$.

$$\begin{aligned} x_{n_1} > x_{n_2} &\quad \text{since } n_1 \in S \text{ and } n_2 > n_1 \\ x_{n_2} > x_{n_3} &\quad \text{since } n_2 \in S \text{ and } n_3 > n_2 \\ &\quad \vdots \\ x_{n_k} > x_{n_{k+1}} &\quad \text{since } n_k \in S \text{ and } n_{k+1} > n_k \\ &\quad \vdots \end{aligned}$$

so $\{x_{n_k}\}$ is a strictly decreasing subsequence of $\{x_n\}$.

If S is finite and nonempty, let $n_1 = (\max S) + 1$; if $S = \emptyset$, let $n_1 = 1$. Then

$$\begin{aligned} n_1 \notin S &\quad \text{so} \quad \exists n_2 > n_1 \text{ s.t. } x_{n_2} \geq x_{n_1} \\ n_2 \notin S &\quad \text{so} \quad \exists n_3 > n_2 \text{ s.t. } x_{n_3} \geq x_{n_2} \\ &\quad \vdots \\ n_k \notin S &\quad \text{so} \quad \exists n_{k+1} > n_k \text{ s.t. } x_{n_{k+1}} \geq x_{n_k} \\ &\quad \vdots \end{aligned}$$

$$\{-1, 3, -10, 7, -, 15\}$$

$$x_n = n \qquad n \geq 10$$

$$x_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ 1 - \frac{1}{n} & n \text{ even} \end{cases}$$

so $\{x_{n_k}\}$ is a (weakly) increasing subsequence of $\{x_n\}$. $\square$

Bolzano-Weierstrass Theorem

Theorem 9 (Thm. 3.3, Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers contains a convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence of real numbers. By the Rising Sun Lemma, find an increasing or decreasing subsequence $\{x_{n_k}\}$. If $\{x_{n_k}\}$ is increasing, then by Theorem 3.1',

$$\lim x_{n_k} = \sup\{x_{n_k} : k \in \mathbf{N}\} \leq \sup\{x_n : n \in \mathbf{N}\} < \infty$$

since the sequence is bounded; since the limit is finite, the subsequence converges. Similarly, if the subsequence is decreasing, it converges. $\square$

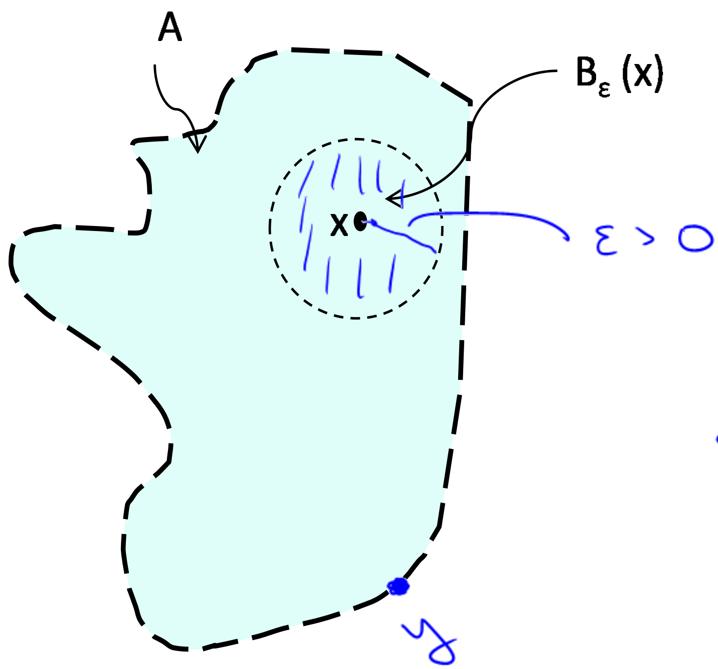
Open and Closed Sets

Definition 1. *Let* (X, d) *be a metric space. A set* $A \subseteq X$ *is* open *if*

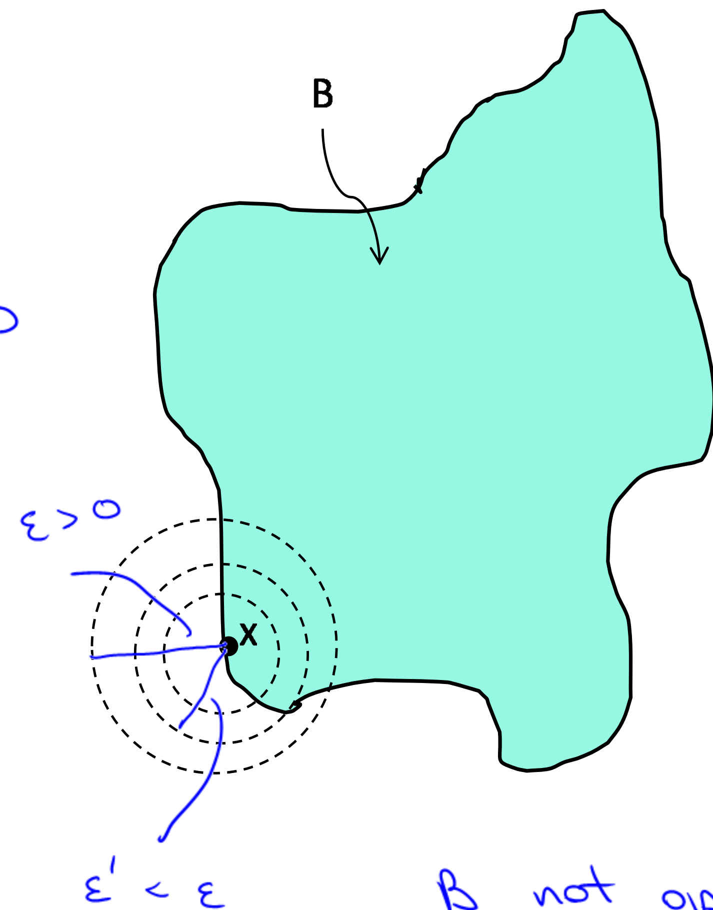
$$\forall x \in A \; \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A$$

A set $C \subseteq X$ *is* closed *if* $X \setminus C$ *is open.*

$$\{y \in X : d(y,x) < \varepsilon\}$$



A open $\quad y \notin A$



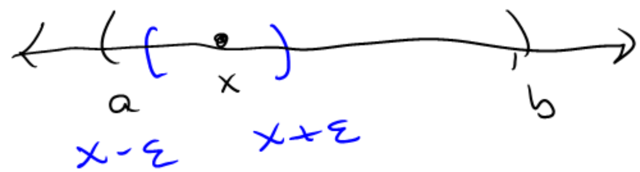
B not open:

$\nexists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq B$

Open and Closed Sets

Example: (a,b) is open in the metric space $\mathbf{E}^1$ ($\mathbf{R}$ with the usual Euclidean metric). Given $x \in (a,b)$, $a < x < b$. Let

$$\varepsilon = \min\{x-a, b-x\} > 0$$



$$-\varepsilon \geq -(x-a)$$
$$\varepsilon \leq b-x$$

Then

$$\begin{aligned} y \in B_\varepsilon(x) &\Rightarrow y \in (x-\varepsilon, x+\varepsilon) \\ &\subseteq (x-(x-a), x+(b-x)) \\ &= (a,b) \end{aligned}$$

so $B_\varepsilon(x) \subseteq (a,b)$, so (a,b) is open.

Notice that ε depends on x; in particular, ε gets smaller as x nears the boundary of the set.

Open and Closed Sets

Example: In $\mathbf{E}^1$, $[a,b]$ is closed. $\mathbf{R} \setminus [a,b] = (-\infty, a) \cup (b, \infty)$ is a union of two open sets, which must be open.

with standard metric

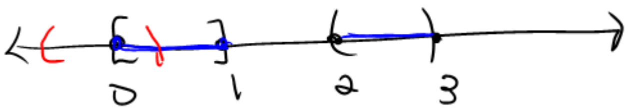
Example: In the metric space $X = [0,1]$, $[0,1]$ is open. With $[0,1]$ as the underlying metric space,

$$\varepsilon \in (0,1): \; B_\varepsilon(0) = \{x \in \underset{X}{[0,1]} : |x - 0| < \varepsilon\} = [0, \varepsilon) \quad \subseteq [0,1]$$

Thus, openness and closedness depend on the underlying metric space as well as on the set.

Open and Closed Sets

Example: Most sets are neither open nor closed. For example, in $\mathbf{E}^1$, $[0,1] \cup (2,3)$ is neither open nor closed.



Example: An open set may consist of a single point. For example, if $X = \mathbf{N}$ and $d(m,n) = |m-n|$, then

$$B_{1/2}(1) = \{m \in \mathbf{N} : |m-1| < 1/2\} = \{1\}$$

Since 1 is the only element of the set $\{1\}$ and $B_{1/2}(1) = \{1\} \subseteq \{1\}$, the set $\{1\}$ is open.

Open and Closed Sets

Example: In any metric space (X, d) both $\emptyset$ and X are open, and both $\emptyset$ and X are closed.

To see that $\emptyset$ is open, note that the statement

$$\forall x \in \emptyset \ \exists \varepsilon > 0 \ B_\varepsilon(x) \subseteq \emptyset$$

is vacuously true since there aren't any $x \in \emptyset$. To see that X is open, note that since $B_\varepsilon(x)$ is by definition $\{z \in X : d(z, x) < \varepsilon\}$, it is trivially contained in X.

Since $\emptyset$ is open, X is closed; since X is open, $\emptyset$ is closed.

$X \setminus \emptyset$ $\qquad$ $X \setminus X$

Open and Closed Sets

Example: Open balls are open sets.

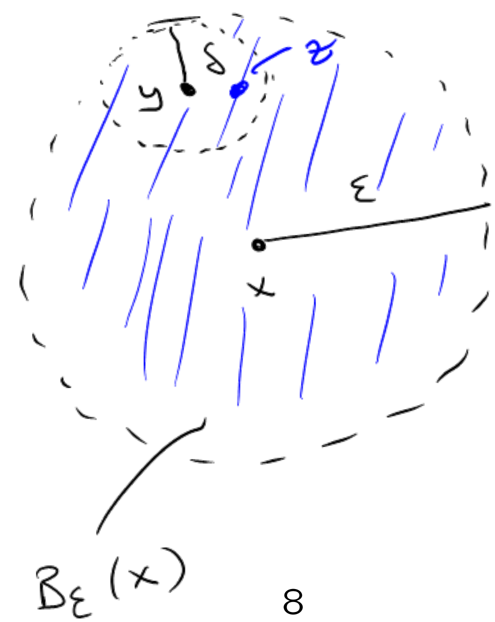
Fix $x \in X$, $\varepsilon > 0$. $B_\varepsilon(x)$ is open:

Suppose $y \in B_\varepsilon(x)$. Then $d(x,y) < \varepsilon$. Let $\delta = \varepsilon - d(x,y) > 0$. If $d(z,y) < \delta$, then

$\Updownarrow$

$z \in B_\delta(y)$

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< \delta + d(x,y) \\ &= \varepsilon - d(x,y) + d(x,y) \\ &= \varepsilon \end{aligned}$$



so $B_\delta(y) \subseteq B_\epsilon(x)$, so $B_\varepsilon(x)$ is open.

Open and Closed Sets

Theorem 1 (Thm. 4.2)**.** *Let* (X, d) *be a metric space. Then*

1. *$\emptyset$ and X are both open, and both closed.*

2. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*

3. *The intersection of a finite collection of open sets is open.*

Proof. 1. We have already shown this.

2. Suppose $\{A_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets.

$$x \in \bigcup_{\lambda \in \Lambda} A_\lambda \;\Rightarrow\; \exists \lambda_0 \in \Lambda \text{ s.t. } x \in A_{\lambda_0} \longleftarrow \text{open}$$

$$\Rightarrow\; \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$$

so $\cup_{\lambda \in \Lambda} A_\lambda$ is open.

3. Suppose $A_1, \ldots, A_n \subseteq X$ are open sets. If $x \in \cap_{i=1}^n A_i$, then

$$x \in \underset{\substack{\uparrow \\ \text{open}}}{A_1}, x \in \underset{\substack{\uparrow \\ \text{open}}}{A_2}, \ldots, x \in \underset{\substack{\uparrow \\ \text{open}}}{A_n}$$

so

$$\exists \varepsilon_1 > 0, \ldots, \varepsilon_n > 0 \text{ s.t. } B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_{\varepsilon_n}(x) \subseteq A_n$$

Let*

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$$

Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n$$

so

$$B_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} A_i$$

which proves that $\cap_{i=1}^{n} A_i$ is open.

$\square$

*Note this is where we need the fact that we are taking a finite intersection. The infimum of an infinite set of positive numbers could be zero. And the intersection of an infinite collection of open sets need not be open.

Interior, Closure, Exterior and Boundary

Definition 2. • *The* interior *of* A*, denoted* $\operatorname{int} A$*, is the largest open set contained in* A *(the union of all open sets contained in* A*).*

A not open $\iff$ $\operatorname{int} A \subsetneq A$

• *The* closure *of* A*, denoted* $\bar{A}$*, is the smallest closed set containing* A *(the intersection of all closed sets containing* A*)*

A not closed $\iff$ $A \subsetneq \bar{A}$

• *The* exterior *of* A*, denoted* $\operatorname{ext} A$*, is the largest open set contained in* $X \setminus A$*.*

$(= \operatorname{int}(X \setminus A))$

• *The* boundary *of* A*, denoted* $\partial A = \overline{(X \setminus A)} \cap \bar{A}$

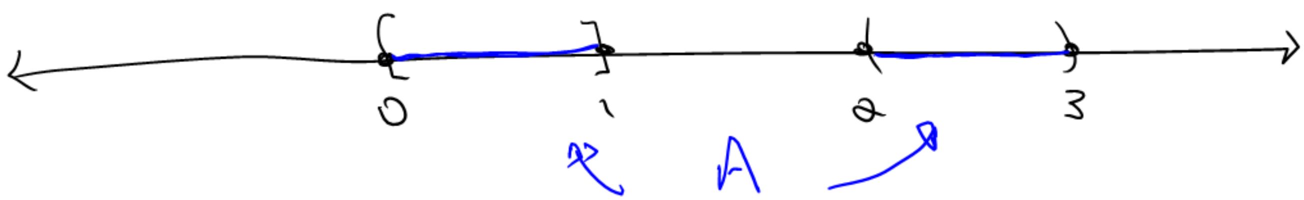
$(= \bar{A} \setminus \operatorname{int} A)$

Interior, Closure, Exterior and Boundary

$\mathbb{R}$ with standard metric:

Example: Let $A = [0, 1] \cup (2, 3)$. Then

$$
\begin{aligned}
\operatorname{int} A &= (0,1) \cup (2,3) \\
\bar{A} &= [0,1] \cup [2,3] \\
\operatorname{ext} A &= \operatorname{int}(X \setminus A) \\
&= (-\infty,0) \cup (1,2) \cup (3,+\infty) \\
\partial A &= \overline{(X \setminus A)} \cap \bar{A} \\
&= \big((-\infty,0] \cup [1,2] \cup [3,+\infty)\big) \cap \big([0,1] \cup [2,3]\big) \\
&= \{0,1,2,3\}
\end{aligned}
$$



Sequences and Closed Sets

Theorem 2 (Thm. 4.13)**.** *A set A in a metric space (X, d) is closed if and only if*

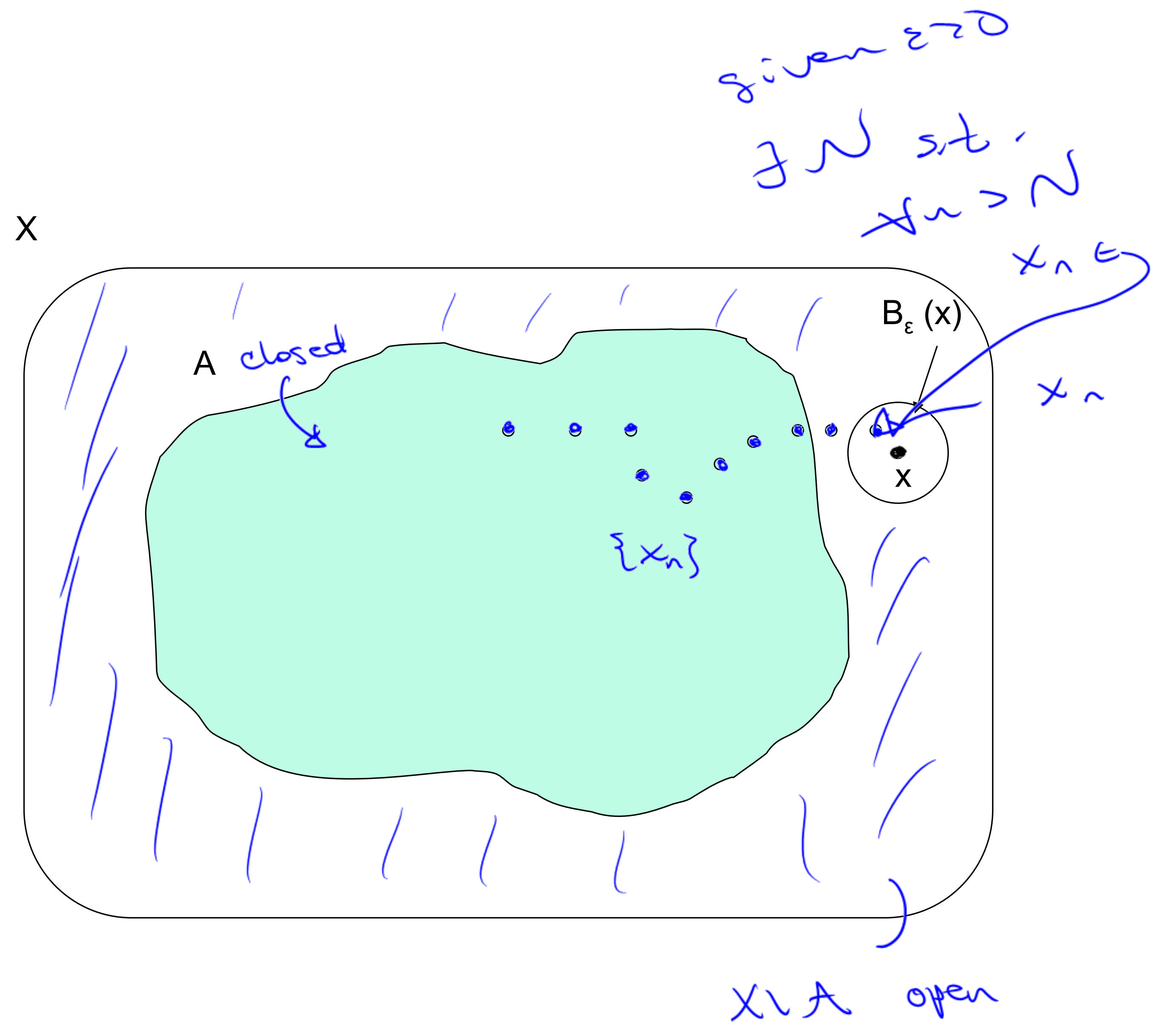
$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

$\Rightarrow$: *Proof.* Suppose A is closed. Then $X \setminus A$ is open. Consider a convergent sequence $x_n \to x \in X$, with $x_n \in A$ for all n. If $x \notin A$, $x \in X \setminus A$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus A$ (why?). Since $x_n \to x$, there exists $N(\varepsilon)$ such that

$$\begin{aligned} n > N(\varepsilon) &\Rightarrow x_n \in B_\varepsilon(x) \\ &\Rightarrow x_n \in X \setminus A \\ &\Rightarrow x_n \notin A \end{aligned}$$

contradiction. Therefore,

$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$



given ε > 0
∃ N s.t.
∀n > N
xₙ ∈
X
Bε (x)
A
closed
xₙ
x
{xₙ}
X \ A open

$\Leftarrow$: Conversely, suppose

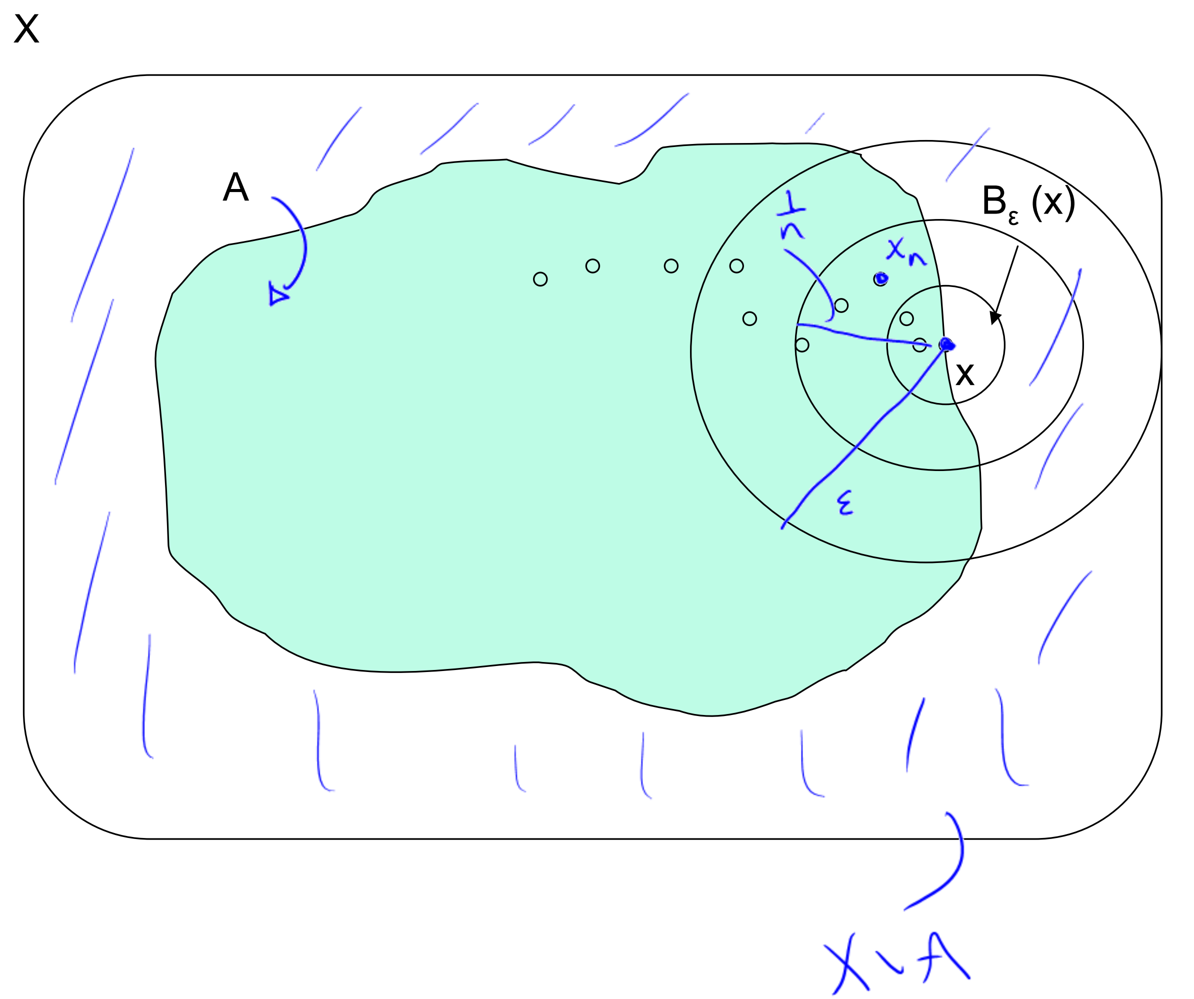
$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

We need to show that A is closed, i.e. $X \setminus A$ is open. Suppose not, so $X \setminus A$ is not open. Then there exists $x \in X \setminus A$ such that for every $\varepsilon > 0$,

$$B_\varepsilon(x) \not\subseteq X \setminus A$$

so there exists $y \in B_\varepsilon(x)$ such that $y \notin X \setminus A$. Then $y \in A$, hence

$$B_\varepsilon(x) \bigcap A \neq \emptyset \qquad \forall \varepsilon > 0$$



$x \in X \setminus A$

Construct a sequence $\{x_n\}$ as follows: for each n, choose

$$x_n \in B_{\frac{1}{n}}(x) \cap A$$

Given $\varepsilon > 0$, we can find $N(\varepsilon)$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$ by the Archimedean Property. So $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$ and $x_n \in B_{\frac{1}{n}}(x) \subseteq B_\varepsilon(x)$. Thus $x_n \to x$. Then $\{x_n\} \subseteq A$, $x_n \to x$, so $x \in A$, contradiction. Therefore, $X \setminus A$ is open, so A is closed. $\square$

Continuity in Metric Spaces

Definition 3. *Let* (X, d) *and* (Y, ρ) *be metric spaces. A function* $f : X \to Y$ *is* continuous at a point $x_0 \in X$ *if*

$$\forall \varepsilon > 0 \ \exists \delta(x_0, \varepsilon) > 0 \ s.t. \ d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

f *is* continuous *if it is continuous at every element of its domain.*

Note that δ can depend on x_0 and ε.

Continuity in Metric Spaces

Continuity at x_0 requires:

- $f(x_0)$ is defined; and
- either
  - x_0 is an isolated point of X, i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x_0) = \{x_0\}$; or

    * read dlf
  - $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$

Continuity in Metric Spaces

Suppose $f : X \to Y$ and $A \subseteq Y$. Define

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$

Theorem 3 (Theorem 6.14)**.** *Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$. Then f is continuous if and only if*

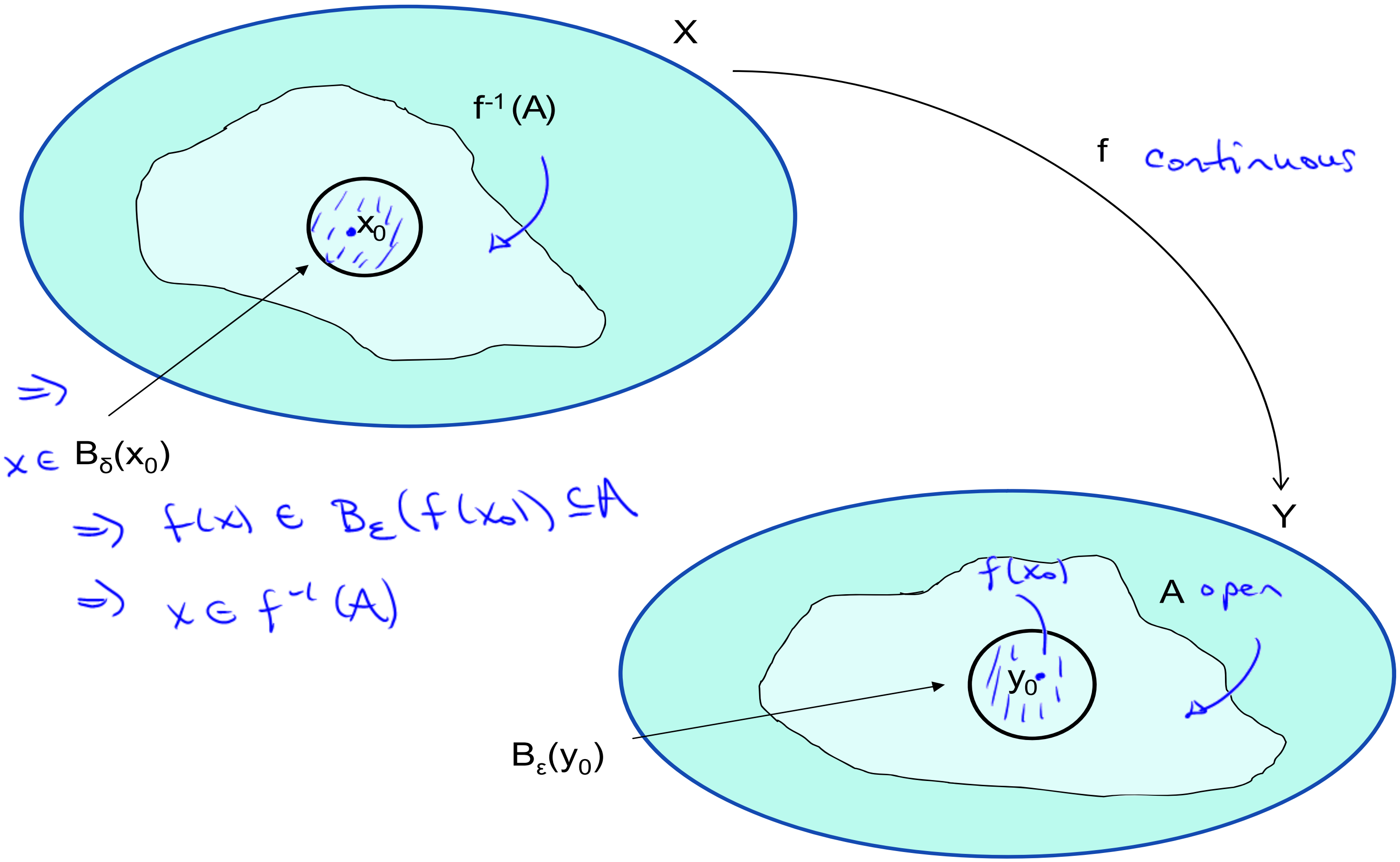
$$f^{-1}(A) \textit{ is open in } X \ \forall A \subseteq Y \textit{ s.t. } A \textit{ is open in } Y$$

Alternatively, f is continuous $\iff$ $f^{-1}(C)$ is closed in X for every closed $C \subseteq Y$.

$\Rightarrow$: *Proof.* Suppose f is continuous. Given $A \subseteq Y$, A open, we must show that $f^{-1}(A)$ is open in X. Suppose $x_0 \in f^{-1}(A)$. Let $y_0 = f(x_0) \in A$. Since A is open, we can find $\varepsilon > 0$ such that $B_\varepsilon(y_0) \subseteq A$. Since f is continuous, there exists $\delta > 0$ such that

$$
\begin{aligned}
x \in B_\delta(x_0) \Rightarrow \quad d(x, x_0) < \delta \;&\Rightarrow\; \rho(f(x), \overbrace{f(x_0)}^{= y_0}) < \varepsilon \\
&\Rightarrow\; f(x) \in B_\varepsilon(y_0) \quad \subseteq A \\
&\Rightarrow\; f(x) \in A \\
&\Rightarrow\; x \in f^{-1}(A)
\end{aligned}
$$

so $B_\delta(x_0) \subseteq f^{-1}(A)$, so $f^{-1}(A)$ is open.



f cont $\Rightarrow$

$\exists\, \delta > 0$ s.t. $x \in B_\delta(x_0)$

$\Rightarrow f(x) \in B_\varepsilon(f(x_0)) \subseteq A$

$\Rightarrow x \in f^{-1}(A)$

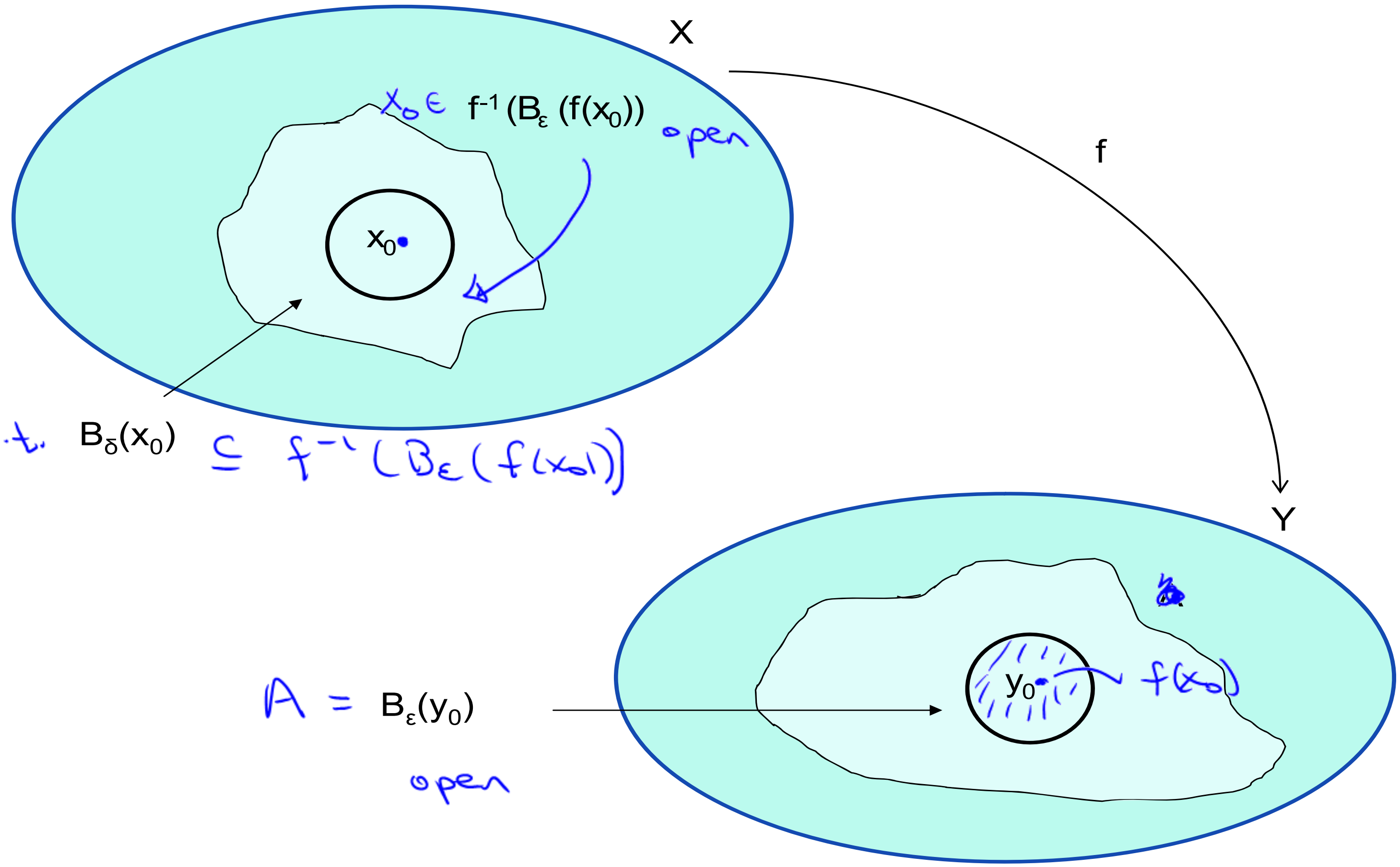
$\Leftarrow$: Conversely, suppose

$$f^{-1}(A) \text{ is open in } X \ \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$

We need to show that f is continuous. Let $x_0 \in X$, $\varepsilon > 0$. Let $A = B_\varepsilon(f(x_0))$. A is an open ball, hence an open set, so $f^{-1}(A)$ is open in X. $x_0 \in f^{-1}(A)$, so there exists $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(A)$.

$$\begin{aligned} d(x, x_0) < \delta \ &\Rightarrow \ x \in B_\delta(x_0) \\ &\Rightarrow \ x \in f^{-1}(A) \\ &\Rightarrow \ f(x) \in A(= B_\varepsilon(f(x_0))) \\ &\Rightarrow \ \rho(f(x), f(x_0)) < \varepsilon \end{aligned}$$

Fix $x_0 \in X$, $\varepsilon > 0$



$\Rightarrow \exists\, \delta > 0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$

$A = B_\varepsilon(y_0)$

open

Thus, we have shown that f is continuous at x_0; since x_0 is an arbitrary point in X, f is continuous. $\square$

Continuity in Metric Spaces

The composition of continuous functions is continuous:

Theorem 4 (Slightly weaker version of Thm. 6.10)**.** *Let* (X, d_X)*,* (Y, d_Y) *and* (Z, d_Z) *be metric spaces. If* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then* $g \circ f : X \to Z$ *is continuous.*

Proof. Suppose $A \subseteq Z$ is open. Since g is continuous, $g^{-1}(A)$ is open in Y; since f is continuous, $f^{-1}(\underbrace{g^{-1}(A)}_{\text{open}})$ is open in X.

We claim that

$$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$$

Observe

$$\begin{aligned} x \in f^{-1}(g^{-1}(A)) &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow g(f(x)) \in A \\ &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow x \in (g \circ f)^{-1}(A) \end{aligned}$$

which establishes the claim. This shows that $(g \circ f)^{-1}(A)$ is open in X, so $g \circ f$ is continuous. $\square$

Uniform Continuity

Definition 4 (Uniform Continuity)**.** *Let* (X, d) *and* (Y, ρ) *be metric spaces. A function* $f : X \to Y$ *is* uniformly continuous *if*

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \textit{s.t.} \ \forall x_0 \in X, \ d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Notice the important contrast with continuity: f is continuous means

$$\forall x_0 \in X, \varepsilon > 0 \ \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Uniform Continuity

Example: Consider $f : (0,1] \to \mathbf{R}$ given by

$$f(x) = \frac{1}{x}, \;\; x \in (0,1]$$

f is continuous (why?). We will show that f is **not** uniformly continuous.

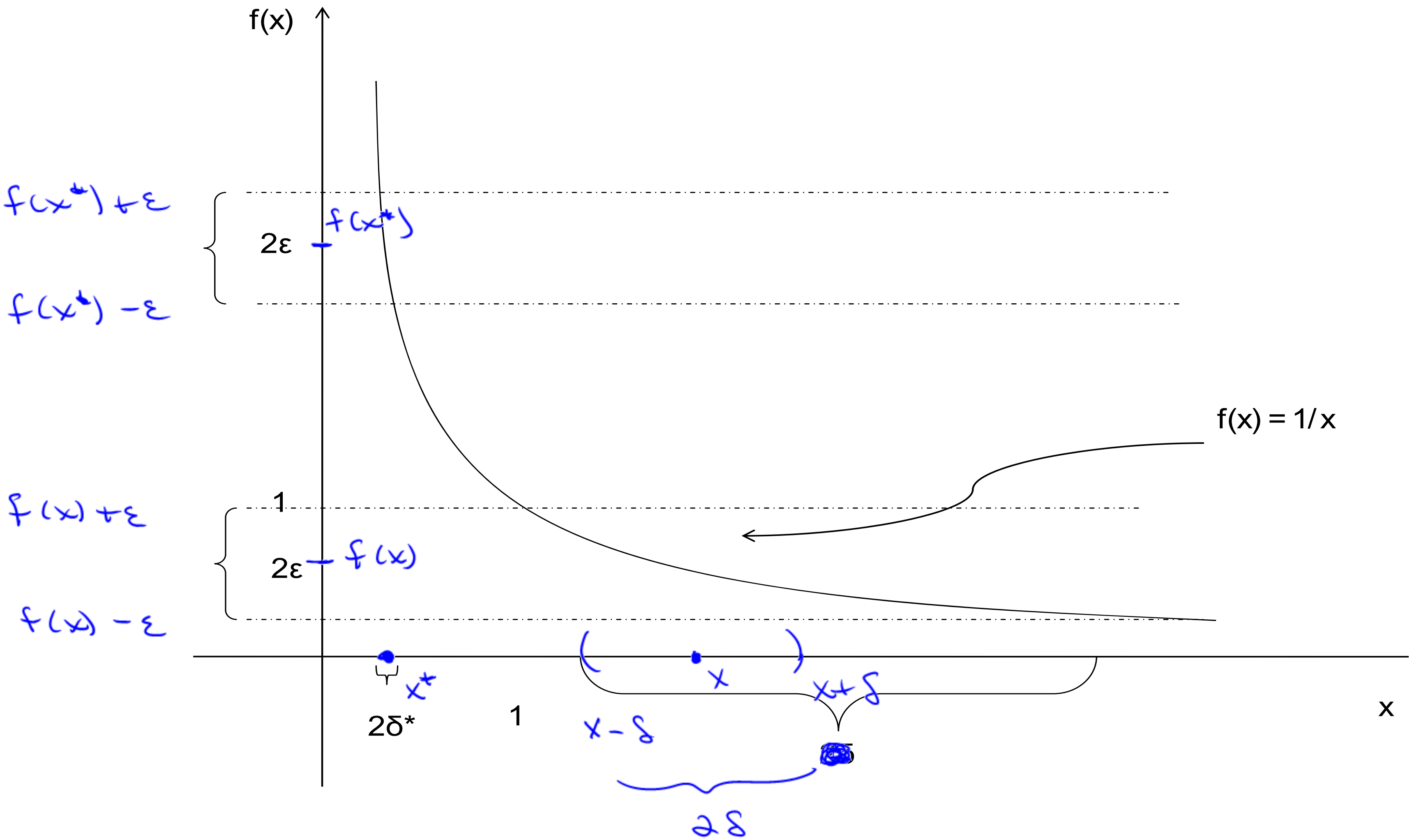
Let $\varepsilon_0 = 1$. Take any $\delta > 0$ with $\delta \leq 1$. Set $x = \frac{\delta}{3}$ and $y = \frac{\delta}{6}$. So

$$|x - y| = \frac{\delta}{6} < \delta$$

But

$$\begin{aligned} |\underbrace{f(x)}_{=\frac{1}{x}} - \underbrace{f(y)}_{=\frac{1}{y}}| &= \frac{|x-y|}{|xy|} = \left|\frac{\delta/6}{\delta^2/18}\right| \\ &= \frac{3}{\delta} > 1 = \varepsilon_0 \end{aligned}$$

Fix $\varepsilon > 0$.



Uniform Continuity

Example: If $f : \mathbf{R} \to \mathbf{R}$ and $f'(x)$ is defined and ~~uniformly~~ bounded on an interval $[a, b]$, then f is uniformly continuous on $[a, b]$. However, even a function with an unbounded derivative may be uniformly continuous. Consider

$$f(x) = \sqrt{x}, \;\; x \in [0, 1]$$

f is continuous (why?). We will show that f is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Then given any $x_0 \in [0, 1]$,

$|x - x_0| < \delta$ implies by the Fundamental Theorem of Calculus

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \int_{x_0}^{x} \frac{1}{2\sqrt{t}}\, dt \right| \\ &\leq \int_0^{|x-x_0|} \frac{1}{2\sqrt{t}}\, dt \\ &= \sqrt{|x - x_0|} \\ &< \sqrt{\delta} \\ &= \sqrt{\varepsilon^2} \\ &= \varepsilon \end{aligned}$$

Thus, f is uniformly continuous on $[0, 1]$, even though $f'(x) \to \infty$ as $x \to 0$.

Lipschitz Continuity

Definition 5. *Let X, Y be normed vector spaces, $E \subseteq X$. A function $f : X \to Y$ is* Lipschitz on E *if*

$$\exists K > 0 \text{ s.t. } \|f(x) - f(z)\|_Y \leq K\|x - z\|_X \quad \forall x, z \in E$$

f *is* locally Lipschitz *on E if*

$$\forall x_0 \in E \ \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

f Lipschitz $\Rightarrow$ $\exists K > 0$ s.t. $\forall x \neq y$,

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} \leq K$$

Notions of Continuity

Lipschitz continuity is stronger than either continuity or uniform continuity:

$$\text{Lipschitz} \Rightarrow \text{locally Lipschitz} \Rightarrow \text{continuous}$$

$$\text{Lipschitz} \Rightarrow \text{uniformly continuous}$$

$f : \mathbb{R}^m \to \mathbb{R}^n$

Every C^1 function (is locally Lipschitz. (Recall that a function $f : \mathbf{R}^m \to \mathbf{R}^n$ is said to be C^1 if all its first partial derivatives exist and are continuous.)

Homeomorphisms

Definition 6. *Let* (X, d) *and* (Y, ρ) *be metric spaces. A function* $f : X \to Y$ *is called a* homeomorphism *if it is one-to-one, onto, continuous, and its inverse function is continuous.*

Topological properties are invariant under homeomorphism:

Homeomorphisms

Suppose that f is a homeomorphism and $U \subset X$. Let $g = f^{-1}: Y \to X$.

$$
\begin{aligned}
y \in g^{-1}(U) &\Leftrightarrow g(y) \in U \\
&\Leftrightarrow y \in f(U) \\
U \text{ open in } X &\Rightarrow g^{-1}(U) \text{ is open in } (f(X), \rho) \\
&\Rightarrow f(U) \text{ is open in } (f(X), \rho)
\end{aligned}
$$

This says that (X, d) and $\left(f(X), \rho|_{f(X)}\right)$ are identical in terms of properties that can be characterized solely in terms of open sets; such properties are called "topological properties."