Econ 204 2023

Lecture 5

Outline

1. Properties of Real Functions (Sect. 2.6, cont.)
2. Monotonic Functions
3. Cauchy Sequences and Complete Metric Spaces
4. Contraction Mappings
5. Contraction Mapping Theorem

Properties of Real Functions

Here we first study properties of functions from $\mathbf{R}$ to $\mathbf{R}$, making use of the additional structure we have in $\mathbf{R}$ as opposed to general metric spaces.

Let $f : X \to \mathbf{R}$ where $X \subseteq \mathbf{R}$. We say f is *bounded above* if

$$f(X) = \{r \in \mathbf{R} : f(x) = r \text{ for some } x \in X\}$$

is bounded above. Similarly, we say f is *bounded below* if $f(X)$ is bounded below. Finally, f is *bounded* if f is both bounded above and bounded below, that is, if $f(X)$ is a bounded set.

Extreme Value Theorem

Theorem 1 (Thm. 6.23, Extreme Value Theorem)**.** *Let $a, b \in \mathbf{R}$ with $a \leq b$ and let $f : [a, b] \to \mathbf{R}$ be a continuous function. Then f assumes its minimum and maximum on $[a, b]$. That is, if*

$$M = \sup_{t \in [a,b]} f(t) \qquad m = \inf_{t \in [a,b]} f(t)$$

then $\exists t_M, t_m \in [a, b]$ such that $f(t_M) = M$ and $f(t_m) = m$.

Proof. Let

$$M = \sup\{f(t) : t \in [a, b]\}$$

If M is finite, then for each n, we may choose $t_n \in [a, b]$ such that $M \geq f(t_n) \geq M - \frac{1}{n}$ (if we couldn't make such a choice, then $M - \frac{1}{n}$ would be an upper bound and M would not be the supremum). If M is infinite, choose t_n such that $f(t_n) \geq n$. By the Bolzano-Weierstrass Theorem, $\{t_n\}$ contains a convergent subsequence $\{t_{n_k}\}$, with

$$\lim_{k \to \infty} t_{n_k} = t_0 \in [a, b]$$

Since f is continuous,

$$\begin{aligned} f(t_0) &= \lim_{t \to t_0} f(t) \\ &= \lim_{k \to \infty} f\left(t_{n_k}\right) \\ &= M \end{aligned}$$

so M is finite and

$$f(t_0) = M = \sup\{f(t) : t \in [a, b]\}$$

so f attains its maximum and is bounded above.

The argument for the minimum is similar. $\square$

Intemediate Value Theorem Redux

Theorem 2 (Thm. 6.24, Intermediate Value Theorem)**.** *Suppose $f : [a, b] \to \mathbf{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.*

Proof. Let

$$B = \{t \in [a, b] : f(t) < d\}$$

$a \in B$, so $B \neq \emptyset$. By the Supremum Property, $\sup B$ exists and is real so let $c = \sup B$. Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$. We claim that $f(c) = d$.

Let

$$t_n = \min\left\{c + \frac{1}{n}, b\right\} \geq c$$

Either $t_n > c$, in which case $t_n \notin B$, or $t_n = c$, in which case $t_n = b$ so $f(t_n) > d$, so again $t_n \notin B$; in either case, $f(t_n) \geq d$. Since f is continuous at c, $f(c) = \lim_{n\to\infty} f(t_n) \geq d$ (Theorem 3.5 in de la Fuente).

Since $c = \sup B$, we may find $s_n \in B$ such that

$$c \geq s_n \geq c - \frac{1}{n}$$

Since $s_n \in B$, $f(s_n) < d$. Since f is continuous at c, $f(c) = \lim_{n\to\infty} f(s_n) \leq d$ (Theorem 3.5 in de la Fuente).

Since $d \leq f(c) \leq d$, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$. $\square$

Monotonic Functions

Definition 1. *A function* $f : \mathbf{R} \to \mathbf{R}$ *is* monotonically increasing *if*

$$y > x \Rightarrow f(y) \geq f(x)$$

Monotonic functions are very well-behaved...

Monotonic Functions

Theorem 3 (Thm. 6.27)**.** *Let* $a, b \in \mathbf{R}$ *with* $a < b$, *and let* $f : (a, b) \to \mathbf{R}$ *be monotonically increasing. Then the one-sided limits*

$$\begin{aligned} f(t^+) &= \lim_{u \to t^+} f(u) \\ f(t^-) &= \lim_{u \to t^-} f(u) \end{aligned}$$

exist and are real numbers for all $t \in (a, b)$.

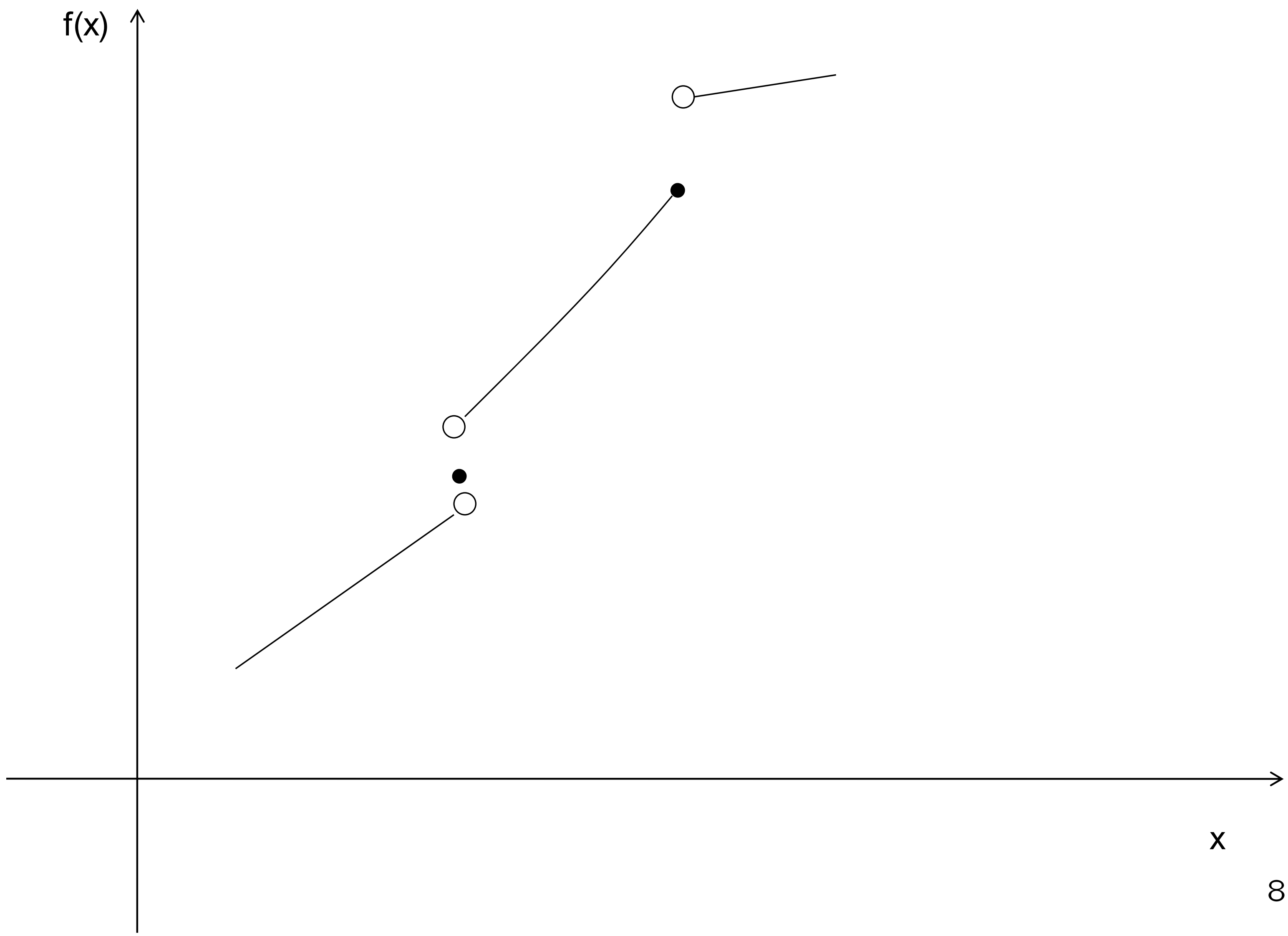
Proof. This is analogous to the proof that a bounded monotone sequence converges. □

Monotonic Functions

We say that f has a *simple jump discontinuity* at t if the one-sided limits $f(t^-)$ and $f(t^+)$ both exist but f is not continuous at t.

Note that there are two ways f can have a simple jump discontinuity at t: either $f(t^+) \neq f(t^-)$, or $f(t^+) = f(t^-) \neq f(t)$.

The previous theorem says that monotonic functions have **only** simple jump discontinuities. Note that monotonicity also implies that $f(t^-) \leq f(t) \leq f(t^+)$. So a monotonic function has a discontinuity at t if and only if $f(t^+) \neq f(t^-)$.


f(x)
x

Monotonic Functions

A monotonic function is continuous "almost everywhere" – except for at most countably many points.

Theorem 4 (Thm. 6.28). *Let* $a, b \in \mathbf{R}$ *with* $a < b$, *and let* $f : (a, b) \to \mathbf{R}$ *be monotonically increasing. Then*

$$D = \{t \in (a, b) : f \text{ is discontinuous at } t\}$$

is finite (possibly empty) or countable.

Proof. If $t \in D$, then $f(t^-) < f(t^+)$ (if the left- and right-hand limits agreed, then by monotonicity they would have to equal $f(t)$, so f would be continuous at t). $\mathbf{Q}$ is dense in $\mathbf{R}$, that is, if

$x, y \in \mathbf{R}$ and $x < y$ then $\exists r \in \mathbf{Q}$ such that $x < r < y$. So for every $t \in D$ we may choose $r(t) \in \mathbf{Q}$ such that

$$f(t^-) < r(t) < f(t^+)$$

This defines a function $r : D \to \mathbf{Q}$. Notice that

$$s > t \Rightarrow f(s^-) \geq f(t^+)$$

so

$$s > t, s, t \in D \Rightarrow r(s) > f(s^-) \geq f(t^+) > r(t)$$

so $r(s) \neq r(t)$. Therefore, r is one-to-one, so it is a bijection from D to a subset of $\mathbf{Q}$. Thus D is finite or countable. $\square$

Cauchy Sequences and Complete Metric Spaces

Roughly, a metric space is complete if "every sequence that ought to converge to a limit has a limit to converge to."

Recall that $x_n \to x$ means

$$\forall \varepsilon > 0 \ \exists N(\varepsilon/2) \text{ s.t. } n > N(\varepsilon/2) \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$$

Observe that if $n, m > N(\varepsilon/2)$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Cauchy Sequences and Complete Metric Spaces

This motivates the following definition:

Definition 2. *A sequence* $\{x_n\}$ *in a metric space* (X, d) *is* Cauchy *if*

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \text{ s.t. } n, m > N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon$$

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to.

Cauchy Sequences and Complete Metric Spaces

Any sequence that **does** converge must be Cauchy:

Theorem 5 (Thm. 7.2)**.** *Every convergent sequence in a metric space is Cauchy.*

Proof. We just did it: Let $x_n \to x$. For every $\varepsilon > 0$ $\exists N$ such that $n > N \Rightarrow d(x_n, x) < \varepsilon/2$. Then

$$m, n > N \Rightarrow d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Example: Let $X = (0, 1]$ and d be the Euclidean metric. Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ in $\mathbf{E}^1$, so $\{x_n\}$ is Cauchy in $\mathbf{E}^1$. Thus $\{x_n\}$ is Cauchy in (X, d). But $\{x_n\}$ does not converge in (X, d).

The Cauchy property depends only on the sequence and the metric d, not on the ambient metric space:

$\{x_n\}$ is Cauchy in (X, d), but $\{x_n\}$ does not **converge** in (X, d) because the point it is trying to converge to (0) is not an element of X.

Complete Metric Spaces and Banach Spaces

Where does every Cauchy sequence get what it wants?

Definition 3. *A metric space* (X, d) *is* complete *if every Cauchy sequence* $\{x_n\} \subseteq X$ *converges to a limit* $x \in X$*.*

Definition 4. *A* Banach space *is a normed vector space that is complete in the metric generated by its norm.*

Complete Metric Spaces and Banach Spaces

Example: Consider the earlier example of $X = (0, 1]$ with d the usual Euclidean metric. The sequence $\{x_n\}$ with $x_n = \frac{1}{n}$ is Cauchy but does not converge, so $((0, 1], d)$ is not complete.

Example: $\mathbf{Q}$ is not complete in the Euclidean metric. To see this, let

$$x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$$

where $\lfloor y \rfloor$ is the greatest integer less than or equal to y; x_n is just equal to the decimal expansion of $\sqrt{2}$ to n digits past the decimal point. Clearly, x_n is rational. $|x_n - \sqrt{2}| \leq 10^{-n}$, so $x_n \to \sqrt{2}$ in $\mathbf{E}^1$, so $\{x_n\}$ is Cauchy in $\mathbf{E}^1$, hence Cauchy in $\mathbf{Q}$; since $\sqrt{2} \notin \mathbf{Q}$, $\{x_n\}$ is not convergent in $\mathbf{Q}$, so $\mathbf{Q}$ is not complete.

Complete Metric Spaces and Banach Spaces

Theorem 6 (Thm. 7.10). $\mathbf{R}$ *is complete with the usual metric (so* $\mathbf{E}^1$ *is a Banach space).*

Proof. Suppose $\{x_n\}$ is a Cauchy sequence in $\mathbf{R}$. Fix $\varepsilon > 0$. Find $N(\varepsilon/2)$ such that

$$n, m > N(\varepsilon/2) \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$$

Let

$$\begin{aligned} \alpha_n &= \sup\{x_k : k \geq n\} \\ \beta_n &= \inf\{x_k : k \geq n\} \end{aligned}$$

Fix $m > N(\varepsilon/2)$. Then

$$\begin{aligned} k \geq m &\Rightarrow k > N(\varepsilon/2) \Rightarrow x_k < x_m + \frac{\varepsilon}{2} \\ &\Rightarrow \alpha_m = \sup\{x_k : k \geq m\} \leq x_m + \frac{\varepsilon}{2} \end{aligned}$$

Since $\alpha_m < \infty$,

$$\limsup x_n = \lim_{n\to\infty} \alpha_n \le \alpha_m \le x_m + \frac{\varepsilon}{2}$$

since the sequence $\{\alpha_n\}$ is decreasing. Similarly,

$$\liminf x_n \ge x_m - \frac{\varepsilon}{2}$$

Therefore,

$$0 \le \limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n \le \varepsilon$$

Since ε is arbitrary,

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n \in \mathbf{R}$$

Thus $\lim_{n\to\infty} x_n$ exists and is real, so $\{x_n\}$ is convergent. $\square$

Complete Metric Spaces and Banach Spaces

Theorem 7 (Thm. 7.11)**.** $\mathbf{E}^n$ *is complete for every* $n \in \mathbf{N}$*.*

Proof. See de la Fuente. $\square$

Complete Metric Spaces and Banach Spaces

Theorem 8 (Thm. 7.9)**.** *Suppose (X, d) is a complete metric space and $Y \subseteq X$. Then $(Y, d) = (Y, d|_Y)$ is complete if and only if Y is a closed subset of X.*

Proof. Suppose (Y, d) is complete. We need to show that Y is closed. Consider a sequence $\{y_n\} \subseteq Y$ such that $y_n \to_{(X,d)} x \in X$. Then $\{y_n\}$ is Cauchy in X, hence Cauchy in Y; since Y is complete, $y_n \to_{(Y,d)} y$ for some $y \in Y$. Therefore, $y_n \to_{(X,d)} y$. By uniqueness of limits, $y = x$, so $x \in Y$. Thus Y is closed.

Conversely, suppose Y is closed. We need to show that Y is complete. Let $\{y_n\}$ be a Cauchy sequence in Y. Then $\{y_n\}$ is Cauchy in X, hence convergent, so $y_n \to_{(X,d)} x$ for some $x \in X$. Since Y is closed, $x \in Y$, so $y_n \to_{(Y,d)} x \in Y$. Thus Y is complete. $\square$

Complete Metric Spaces and Banach Spaces

Theorem 9 (Thm. 7.12)**.** *Given* $X \subseteq \mathbf{R}^n$, *let* $C(X)$ *be the set of bounded continuous functions from* X *to* $\mathbf{R}$ *with*

$$\|f\|_\infty = \text{sup}\{|f(x)| : x \in X\}$$

Then $C(X)$ *is a Banach space.*

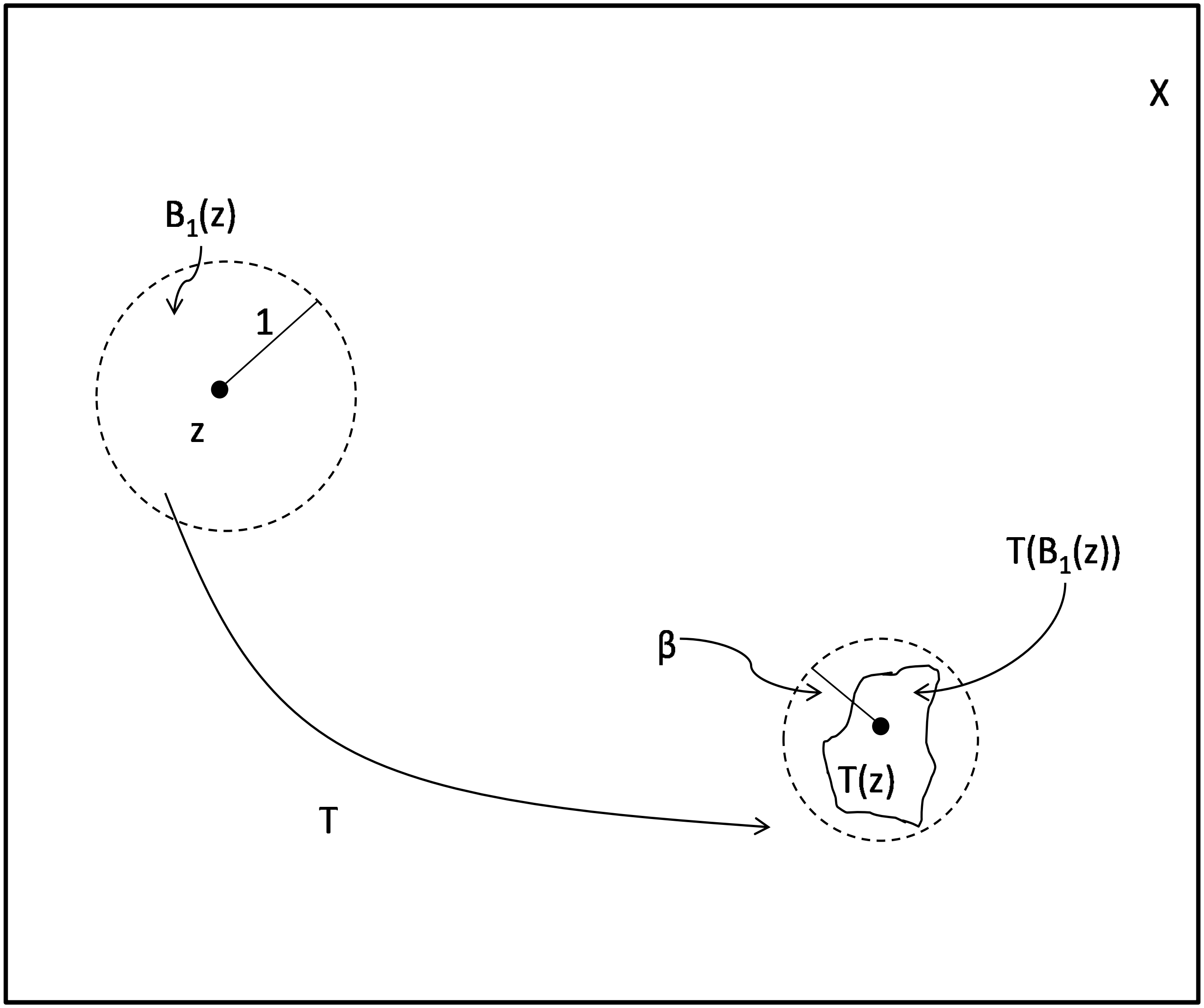
Contractions

Definition 5. *Let* (X, d) *be a nonempty complete metric space. An* operator *is a function* $T : X \to X$*.*

An operator T *is a* contraction of modulus β *if* $0 \leq \beta < 1$ *and*

$$d(T(x), T(y)) \leq \beta d(x, y) \quad \forall x, y \in X$$

A contraction shrinks distances by a **uniform** factor $\beta < 1$.


X
B1(z)
1
z
T(B1(z))
β
T(z)
T

Contractions

Theorem 10. *Every contraction is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{\beta}$. Then $\forall x, y$ such that $d(x, y) < \delta$,

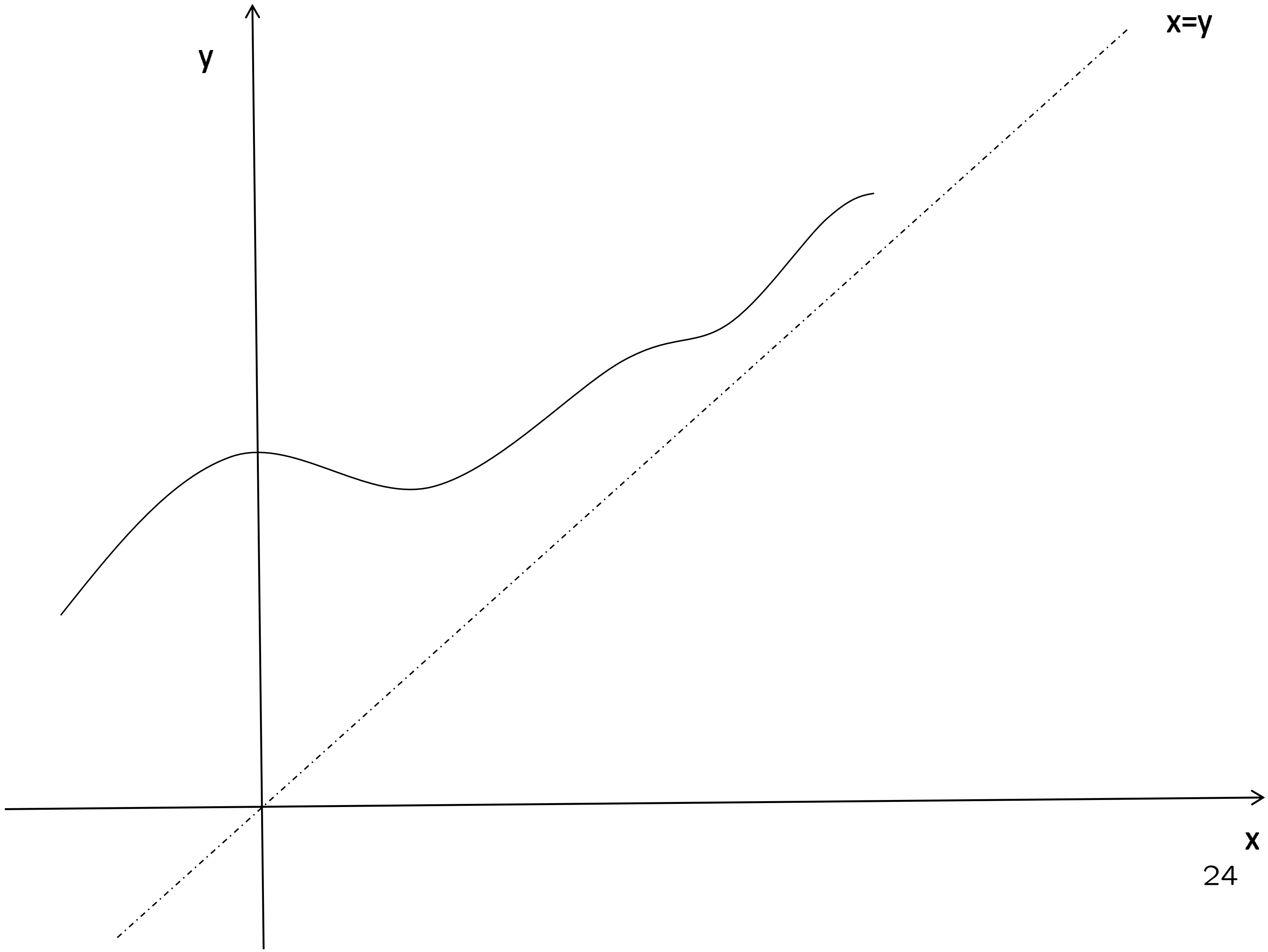
$$d(T(x), T(y)) \leq \beta d(x, y) < \beta\delta = \varepsilon$$

□

Note that a contraction is Lipschitz continuous with Lipschitz constant $\beta < 1$ (and hence also uniformly continuous).

Contractions and Fixed Points

Definition 6. *A* fixed point *of an operator* T *is point* $x^* \in X$ *such that* $T(x^*) = x^*$*.*



y
x=y
x

Contraction Mapping Theorem

Theorem 11 (Thm. 7.16, Contraction Mapping Theorem)**.** *Let* (X, d) *be a nonempty complete metric space and* $T : X \to X$ *a contraction with modulus* $\beta < 1$. *Then*

1. T *has a unique fixed point* x^*.

2. *For every* $x_0 \in X$*, the sequence* $\{x_n\}$ *where*

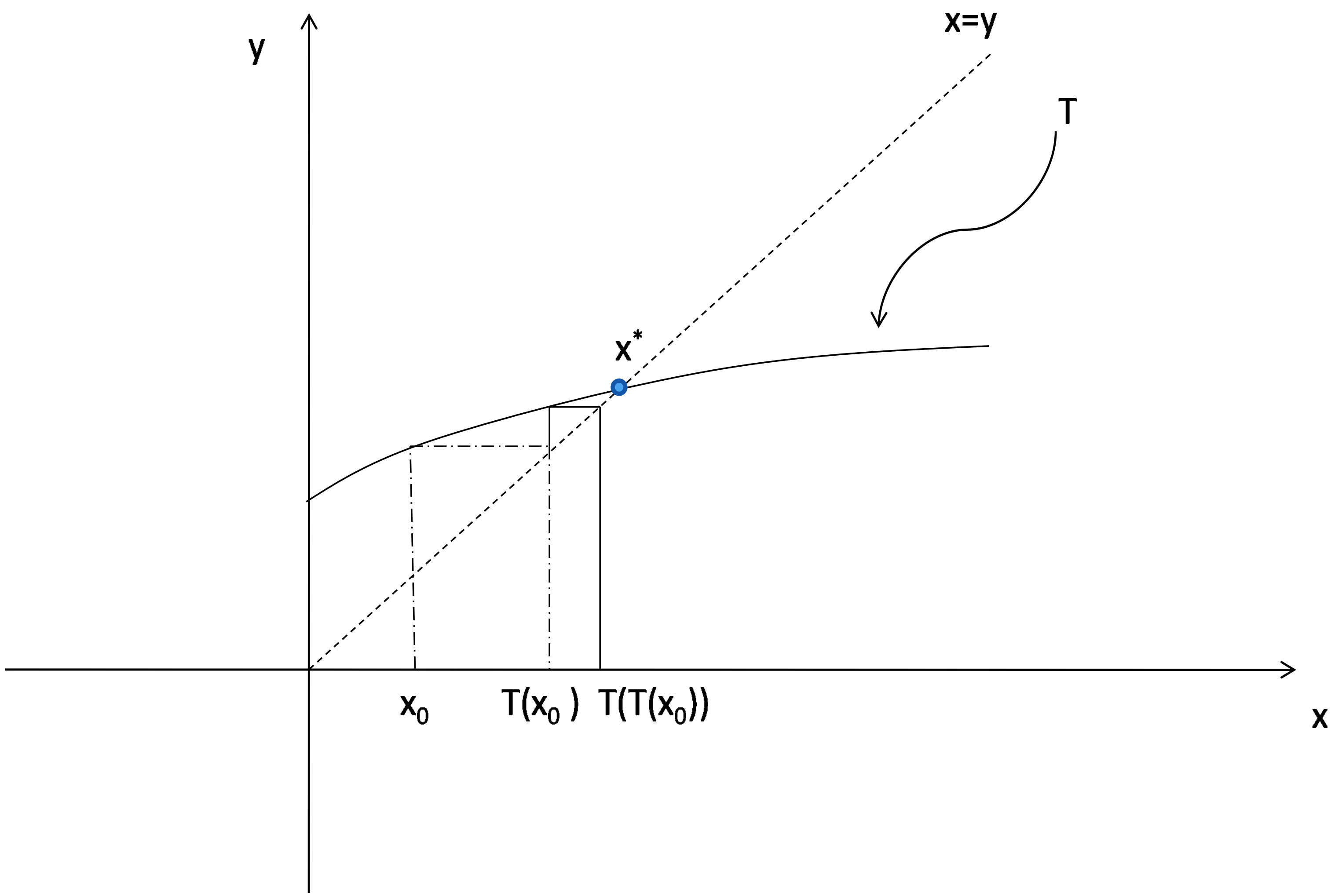
   $x_1 = T(x_0), x_2 = T(x_1) = T(T(x_0)), ..., x_n = T(x_{n-1})$ *for each* n

   converges to x^*.

Note that the theorem asserts both the existence and uniqueness of the fixed point, as well as giving an algorithm to find the fixed point of a contraction.

Also note that the algorithm generates a sequence that converges to the fixed point for any initial point x_0.

Later in the course we will discuss more general fixed point theorems which, in contrast, only guarantee existence, and are not constructive.


y
x=y
T
x*
x
x0
T(x0)
T(T(x0))

Proof. Define the sequence $\{x_n\}$ as above by first fixing $x_0 \in X$ and then letting $x_n = T(x_{n-1}) = T^n(x_0)$ for $n = 1, 2, \ldots$, where $T^n = T \circ T \circ \ldots \circ T$ is the n-fold iteration of T. We first show that $\{x_n\}$ is Cauchy, and hence converges to a limit x. Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \beta d(x_n, x_{n-1}) = \beta d(T(x_{n-1}), T(x_{n-2})) \\ &\leq \beta^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \beta^n d(x_1, x_0) \end{aligned}$$

Then for any $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq (\beta^{n-1} + \beta^{n-2} + \cdots + \beta^m) d(x_1, x_0) \\ &= d(x_1, x_0) \sum_{\ell=m}^{n-1} \beta^\ell \\ &< d(x_1, x_0) \sum_{\ell=m}^{\infty} \beta^\ell \\ &= \frac{\beta^m}{1-\beta} d(x_1, x_0) \quad \text{(sum of a geometric series)} \end{aligned}$$

Fix $\varepsilon > 0$. Since $\frac{\beta^m}{1-\beta} d(x_1, x_0) \to 0$ as $m \to \infty$, choose $N(\varepsilon)$ such that for any $m > N(\varepsilon)$, $\frac{\beta^m}{1-\beta} d(x_1, x_0) < \varepsilon$. Then for $n, m > N(\varepsilon)$,

$$d(x_n, x_m) \leq \frac{\beta^m}{1-\beta} d(x_1, x_0) < \varepsilon$$

Therefore, $\{x_n\}$ is Cauchy. Since (X, d) is complete, $x_n \to x^*$ for some $x^* \in X$.

Next, we show that x^* is a fixed point of T.

$$\begin{aligned} T(x^*) &= T\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} T(x_n) \text{ since } T \text{ is continuous} \\ &= \lim_{n\to\infty} x_{n+1} \\ &= x^* \end{aligned}$$

so x^* is a fixed point of T.

Finally, we show that there is at most one fixed point. Suppose x^* and y^* are both fixed points of T, so $T(x^*) = x^*$ and $T(y^*) = y^*$.

Then

$$
\begin{aligned}
d(x^*, y^*) &= d(T(x^*), T(y^*)) \\
&\leq \beta d(x^*, y^*) \\
\Rightarrow (1 - \beta) d(x^*, y^*) &\leq 0 \\
\Rightarrow d(x^*, y^*) &\leq 0
\end{aligned}
$$

So $d(x^*, y^*) = 0$, which implies $x^* = y^*$. $\square$

Continuous Dependence on Paramters

Theorem 12. (Thm. 7.18', Continuous Dependence on Parameters) *Let (X, d) and (Ω, ρ) be two metric spaces and $T : X \times \Omega \to X$. For each $\omega \in \Omega$ let $T_\omega : X \to X$ be defined by*

$$T_\omega(x) = T(x, \omega)$$

Suppose (X, d) is complete, T is continuous in ω, that is $T(x, \cdot) : \Omega \to X$ is continuous for each $x \in X$, and $\exists \beta < 1$ such that T_ω is a contraction of modulus $\beta \quad \forall \omega \in \Omega$. Then the fixed point function $x^* : \Omega \to X$ defined by

$$x^*(\omega) = T_\omega(x^*(\omega))$$

is continuous.

Blackwell's Sufficient Conditions

An important result due to Blackwell gives a set of sufficient conditions for an operator to be a contraction that is particularly useful in dynamic programming problems.

Let X be a set, and let $B(X)$ be the set of all bounded functions from X to $\mathbf{R}$. Then $(B(X), \|\cdot\|_\infty)$ is a normed vector space.

Notice that below we use shorthand notation that identifies a constant function with its constant value in $\mathbf{R}$, that is, we write interchangeably $a \in \mathbf{R}$ and $a : X \to \mathbf{R}$ to denote the function such that $a(x) = a \ \forall x \in X$.

Blackwell's Sufficient Conditions

Theorem 13. (Blackwell's Sufficient Conditions) *Consider* $B(X)$ *with the sup norm* $\|\cdot\|_\infty$. *Let* $T : B(X) \to B(X)$ *be an operator satisfying*

1. *(monotonicity)* $f(x) \leq g(x)\ \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x)\ \forall x \in X$

2. *(discounting)* $\exists \beta \in (0,1)$ *such that for every* $a \geq 0$ *and* $x \in X$,

$$(T(f+a))(x) \leq (Tf)(x) + \beta a$$

Then T *is a contraction with modulus* β.

Proof. Fix $f, g \in B(X)$. By the definition of the sup norm,

$$f(x) \leq g(x) + \|f - g\|_\infty \ \forall x \in X$$

Then

$$\begin{aligned} (Tf)(x) &\leq (T(g + \|f-g\|_\infty))(x) \quad \forall x \in X \qquad \text{(monotonicity)} \\ &\leq (Tg)(x) + \beta\|f-g\|_\infty \quad \forall x \in X \qquad \text{(discounting)} \end{aligned}$$

Thus

$$(Tf)(x) - (Tg)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X$$

Reversing the roles of f and g above gives

$$(Tg)(x) - (Tf)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X$$

Thus

$$\|T(f) - T(g)\|_\infty \leq \beta\|f - g\|_\infty$$

Thus T is a contraction with modulus β □