Announcements

• PS 2 due Tuesday 8/1

Econ 204 2023

Lecture 6

Outline

1. Open Covers
2. Compactness
3. Sequential Compactness
4. Totally Bounded Sets
5. Heine-Borel Theorem
6. Extreme Value Theorem

Open Covers

Definition 1. *A collection of sets*

$$\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$$

$\subseteq X$

in a metric space (X, d) *is an* open cover *of* A *if* U_λ *is open for all* $\lambda \in \Lambda$ *and*

$$\cup_{\lambda \in \Lambda} U_\lambda \supseteq A$$

Notice that Λ may be finite, countably infinite, or uncountable.

Compactness

Definition 2. *A set A in a metric space is* compact *if every open cover of A contains a finite subcover of A. In other words, if $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A, there exist $n \in \mathbf{N}$ and $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that*

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

This definition does **not** say "A has a finite open cover" (fortunately, since this is vacuous...).

Instead for **any** arbitrary open cover you must specify a finite subcover of this **given** open cover.

Compactness

Example: $(0,1]$ is not compact in $\mathbf{E}^1$. ($\mathbb{R}$ with standard metric)

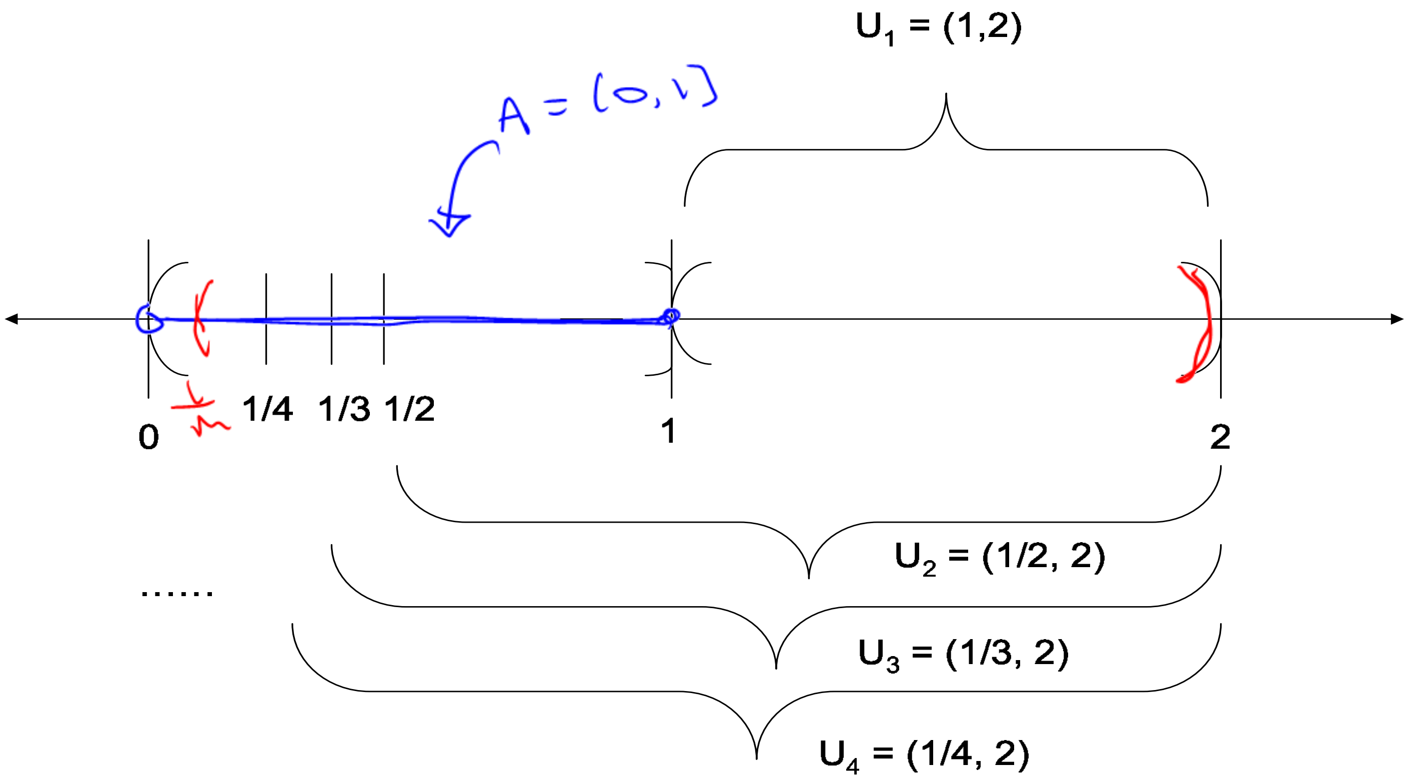
To see this, let

$$\mathcal{U} = \left\{ U_m = \left(\frac{1}{m}, 2 \right) : m \in \mathbf{N} \right\}$$ U_m open $\forall m$

Then

$$\cup_{m \in \mathbf{N}} U_m = (0,2) \supset (0,1]$$

$\Rightarrow \mathcal{U}$ is an open cover of $(0,1]$



$U_1 = (1,2)$
$A = [0,1]$
0
$\frac{1}{n}$
1/4
1/3
1/2
1
2
$U_2 = (1/2, 2)$
……
$U_3 = (1/3, 2)$
$U_4 = (1/4, 2)$

Given any finite subset $\{U_{m_1}, \ldots, U_{m_n}\}$ of $\mathcal{U}$, let

$$m = \max\{m_1, \ldots, m_n\} > 0$$

Then

$$\cup_{i=1}^{n} U_{m_i} = U_m = \left(\frac{1}{m}, 2\right) \not\supseteq (0, 1]$$

So $(0, 1]$ is not compact.

What about $[0, 1]$? This argument doesn't work...

Compactness

Example: $[0, \infty)$ is closed but not compact. (in $\mathbb{R}$ with standard metric)

To see that $[0, \infty)$ is not compact, let

$$\mathcal{U} = \{U_m = (-1, m) : m \in \mathbf{N}\}$$

$\bigcup_{m \in \mathbb{N}} (-1, m) = (-1, \infty) \supseteq [0, \infty)$

$\Rightarrow \mathcal{U}$ open cover of $[0, \infty)$

Given any finite subset

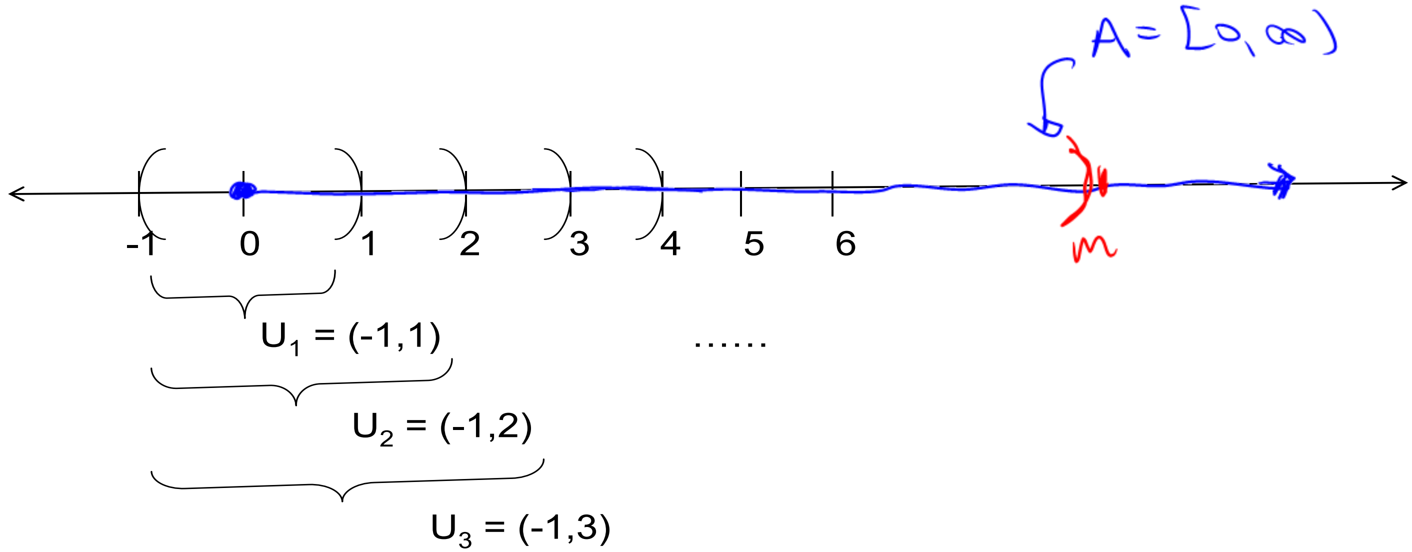
$$\{U_{m_1}, \ldots, U_{m_n}\}$$

of $\mathcal{U}$, let

$$0 < m = \max\{m_1, \ldots, m_n\} < +\infty$$

Then

$$U_{m_1} \cup \cdots \cup U_{m_n} = (-1, m) \not\supseteq [0, \infty)$$


$A = [0, \infty)$
-1
0
1
2
3
4
5
6
m
$U_1 = (-1,1)$
......
$U_2 = (-1,2)$
$U_3 = (-1,3)$

Compactness

Theorem 1 (Thm. 8.14)**.** *Every closed subset A of a compact metric space (X, d) is compact.*

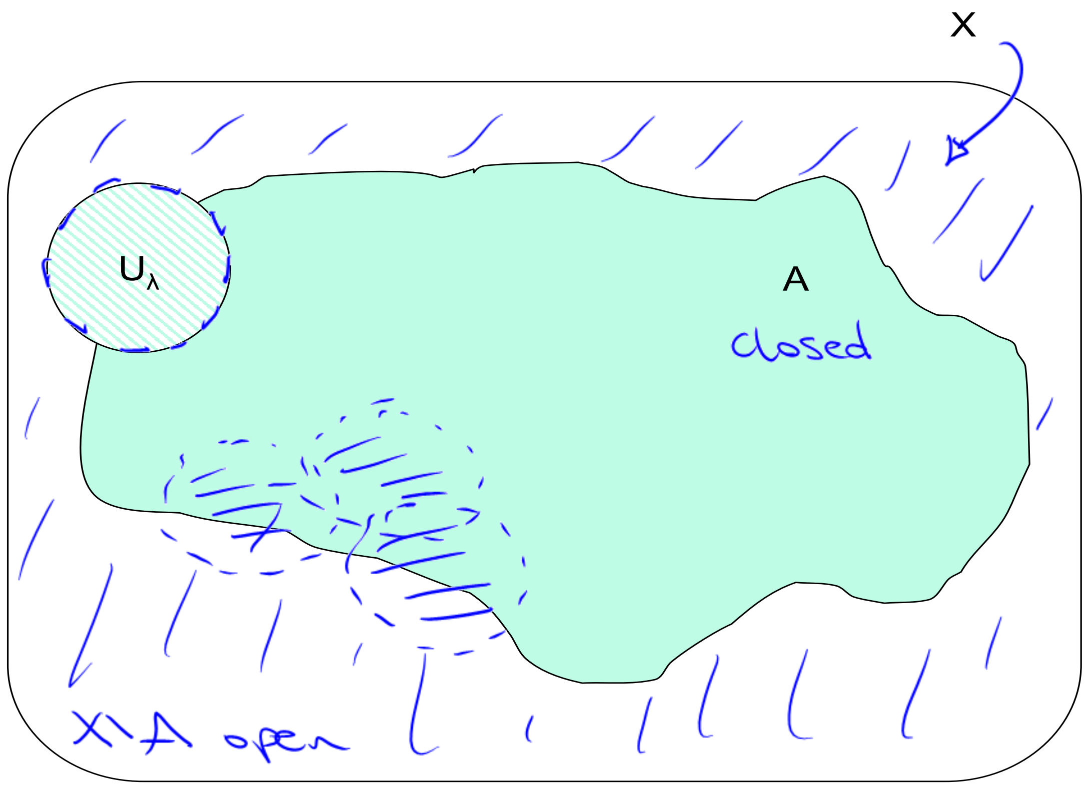
Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A. In order to use the compactness of X, we need to produce an open cover of X. There are two ways to do this:

$$\begin{aligned} U'_\lambda &= U_\lambda \cup (X \setminus A) \\ \Lambda' &= \Lambda \cup \{\lambda_0\},\ U_{\lambda_0} = X \setminus A \end{aligned}$$

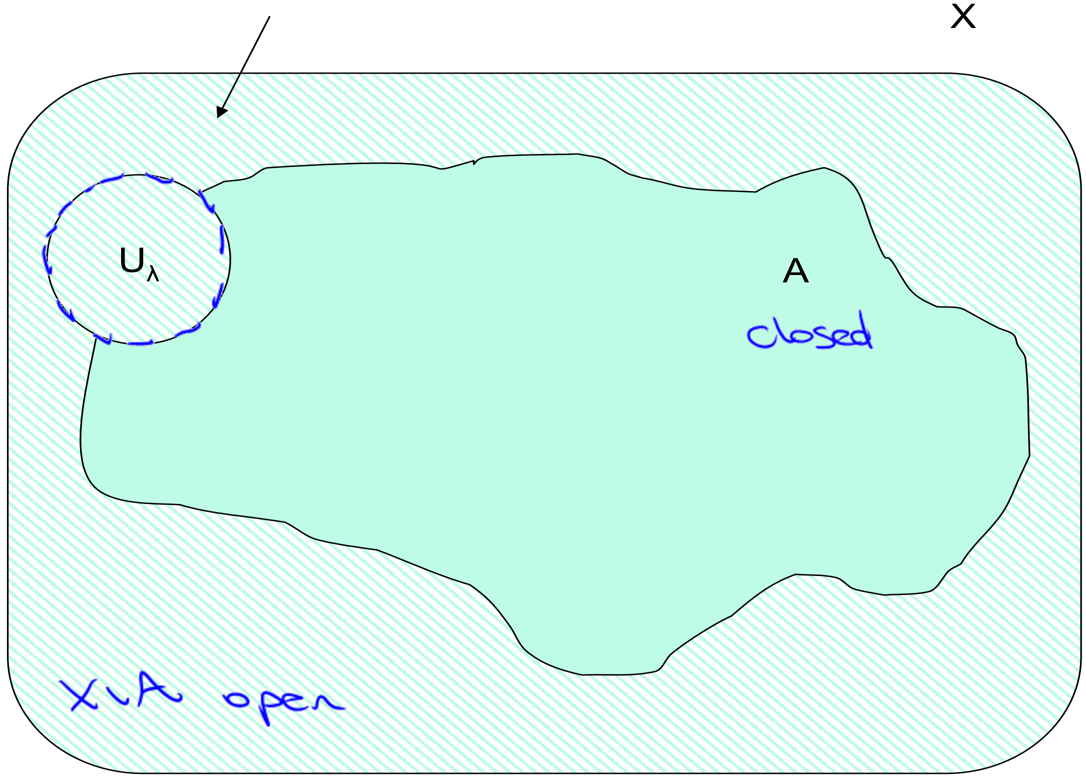
open since A closed

We choose the first path, and let

$$U'_\lambda = U_\lambda \cup (X \setminus A) \qquad \forall\ \lambda \in \Lambda$$


X
U_λ
A
closed
X\A open

$U_\lambda' = U_\lambda \cup (X \setminus A)$



Since A is closed, $X \setminus A$ is open; since U_λ is open, so is U'_λ.

Then $x \in X \Rightarrow x \in A$ or $x \in X \setminus A$. If $x \in A$, $\exists \lambda \in \Lambda$ s.t. $x \in U_\lambda \subseteq U'_\lambda$. If instead $x \in X \setminus A$, then $\forall \lambda \in \Lambda$, $x \in U'_\lambda$. Therefore, $X \subseteq \cup_{\lambda \in \Lambda} U'_\lambda$, so $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of X.

Since X is compact,

$$\exists \lambda_1, \ldots, \lambda_n \in \Lambda \text{ s.t. } X \subseteq U'_{\lambda_1} \cup \cdots \cup U'_{\lambda_n}$$

Then

$$\begin{aligned} a \in A &\Rightarrow a \in X \\ &\Rightarrow a \in U'_{\lambda_i} \text{ for some } i \\ &\Rightarrow a \in U_{\lambda_i} \cup (X \setminus A) \\ &\Rightarrow a \in U_{\lambda_i} \end{aligned}$$

so

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

Thus A is compact. $\square$

Compactness

closed $\not\Rightarrow$ compact, but the converse is true: in any metric space

Theorem 2 (Thm. 8.15). *If A is a compact subset of the metric space (X, d), then A is closed.*

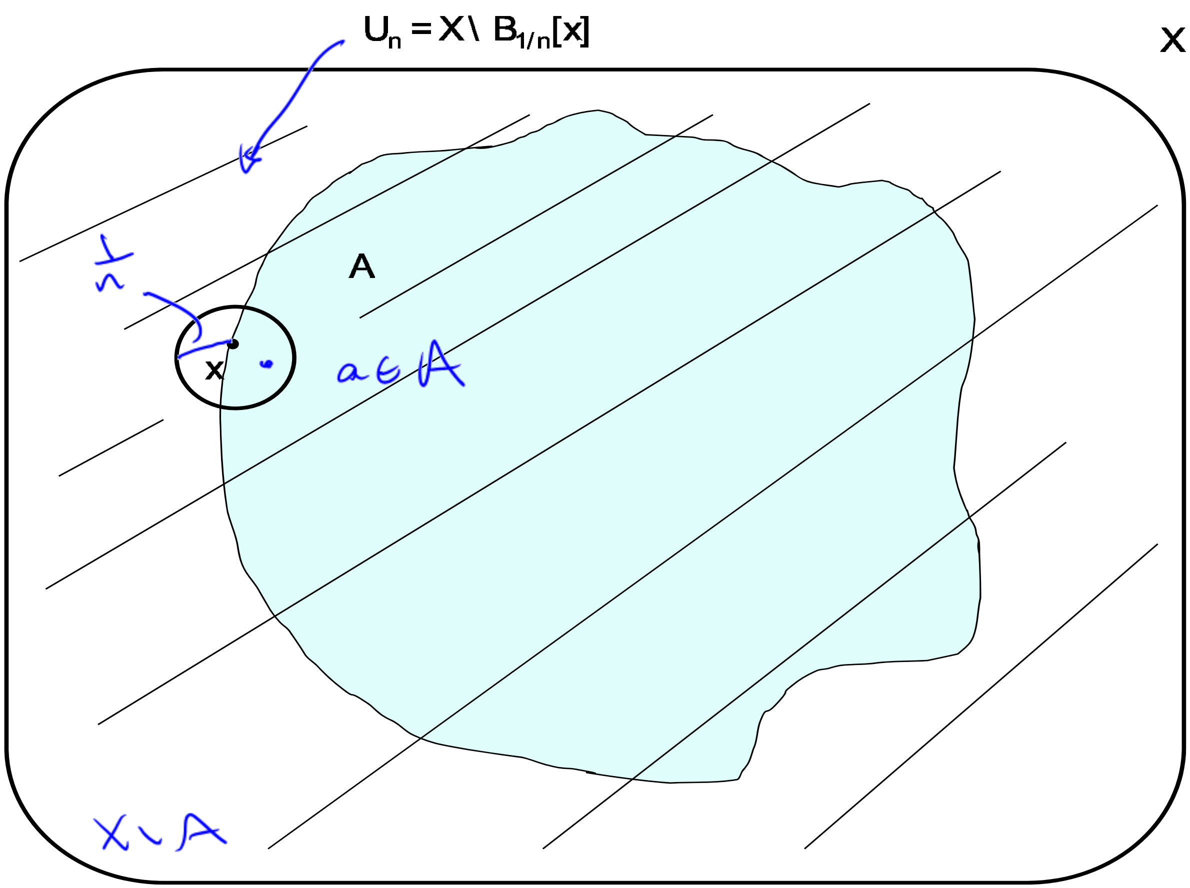
Proof. Suppose by way of contradiction that A is not closed. Then $X \setminus A$ is not open, so we can find a point $x \in X \setminus A$ such that, for every $\varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$, and hence $A \cap B_\varepsilon[x] \neq \emptyset$. For $n \in \mathbf{N}$, let

$\Rightarrow A \not\subseteq X \setminus B_\varepsilon[x]$

$$U_n = X \setminus B_{\frac{1}{n}}[x]$$

open

$\forall n \quad A \cap B_{\frac{1}{n}}[x] \neq \emptyset$



Each U_n is open, and

$$\cup_{n \in \mathbf{N}} U_n = X \setminus \{x\} \supseteq A$$

since $x \notin A$. Therefore, $\{U_n : n \in \mathbf{N}\}$ is an open cover for A. Since A is compact, there is a finite subcover $\{U_{n_1}, \ldots, U_{n_k}\}$. Let $n = \max\{n_1, \ldots, n_k\}$ $< \infty$. Then

$$\begin{aligned}
U_n &= X \setminus B_{\frac{1}{n}}[x] \\
&\supseteq X \setminus B_{\frac{1}{n_j}}[x] = U_{n_j} \quad (j = 1, \ldots, k) \\
\Rightarrow \quad U_n &\supseteq \cup_{j=1}^{k} U_{n_j} \\
&\supseteq A
\end{aligned}$$

But $A \cap B_{\frac{1}{n}}[x] \neq \emptyset$, so $A \not\subseteq X \setminus B_{\frac{1}{n}}[x] = U_n$, a contradiction which proves that A is closed. $\square$

Sequential Compactness

Definition 3. *A set A in a metric space (X, d) is* sequentially compact *if every sequence of elements of A contains a convergent subsequence whose limit lies in A.*

Sequential Compactness

Theorem 3 (Thms. 8.5, 8.11)**.** *A set A in a metric space (X, d) is compact if and only if it is sequentially compact.*

$\Rightarrow$: *Proof.* Suppose A is compact. We will show that A is sequentially compact.

If not, we can find a sequence $\{x_n\}$ of elements of A such that no subsequence converges to **any** element of A. Recall that a is a cluster point of the sequence $\{x_n\}$ means that

$$\forall \varepsilon > 0 \quad \{n : x_n \in B_\varepsilon(a)\} \text{ is infinite}$$

and this is equivalent to the statement that there is a subsequence $\{x_{n_k}\}$ converging to a. Thus, **no** element $a \in A$ can be a cluster point for $\{x_n\}$, and hence

$$\forall a \in A \ \ \exists \varepsilon_a > 0 \text{ s.t. } \{n : x_n \in B_{\varepsilon_a}(a)\} \text{ is finite} \tag{1}$$

Then

$$\{B_{\varepsilon_a}(a) : a \in A\}$$

is an open cover of A (if A is uncountable, it will be an uncountable open cover). Since A is compact, there is a finite subcover

$$\left\{B_{\varepsilon_{a_1}}(a_1), \ldots, B_{\varepsilon_{am}}(a_m)\right\}$$ $A \subseteq B_{\varepsilon_{a_1}}(a_1) \cup \cdots \cup B_{\varepsilon_{a_m}}(a_m)$

Then $\{x_n\} \subseteq A \Rightarrow$

$$\begin{aligned}
\mathbf{N} &= \{n : x_n \in A\} \subseteq \\
&\subseteq \left\{n : x_n \in \left(B_{\varepsilon_{a_1}}(a_1) \cup \cdots \cup B_{\varepsilon_{am}}(a_m)\right)\right\} \\
&= \{n : x_n \in B_{\varepsilon_{a_1}}(a_1)\} \cup \cdots \cup \{n : x_n \in B_{\varepsilon_{am}}(a_m)\}
\end{aligned}$$

so $\mathbf{N}$ is contained in a finite union of sets, each of which is finite by Equation (1). Thus, $\mathbf{N}$ must be finite, a contradiction which proves that A is sequentially compact.

For the converse, see de la Fuente. $\square$

Totally Bounded Sets

Definition 4. *A set A in a metric space (X, d) is* totally bounded *if, for every $\varepsilon > 0$,*

$$\exists x_1, \ldots, x_n \in A \text{ s.t. } A \subseteq \cup_{i=1}^{n} B_\varepsilon(x_i)$$

Recall: $A \subseteq X$ is <u>bounded</u> if $\exists\, \beta > 0$ and $\exists\, x \in X$

s.t. $A \subseteq B_\beta(x)$

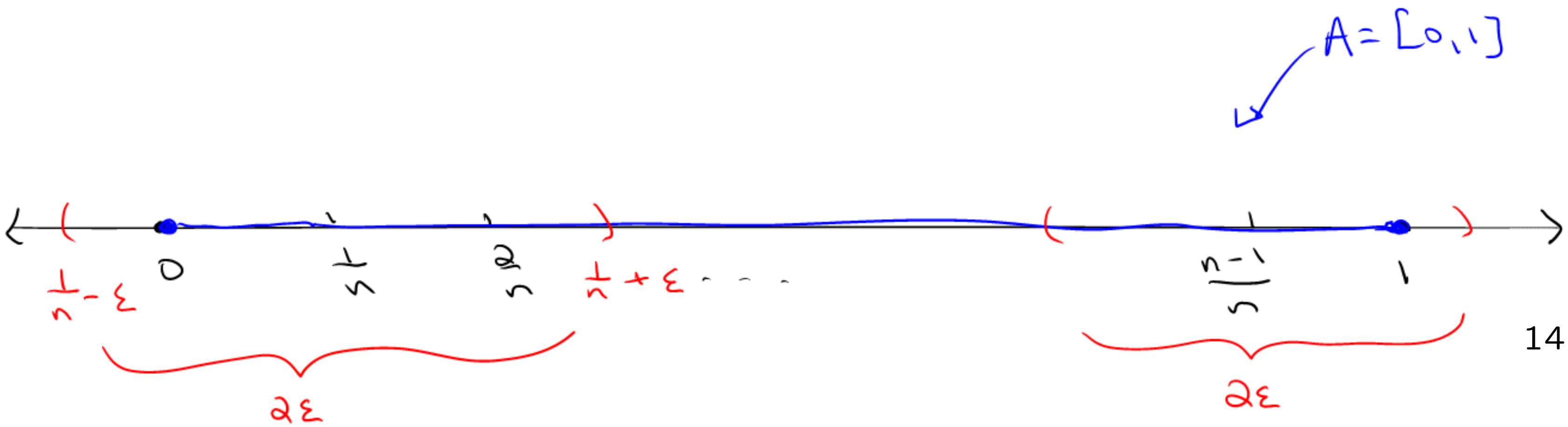
Totally Bounded Sets

Example: Take $A = [0,1]$ $\subseteq \mathbb{R}$ with the Euclidean metric. Given $\varepsilon > 0$, let $n > \frac{1}{\varepsilon}$. Then we may take

$\Rightarrow \varepsilon > \frac{1}{n}$

$$x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}$$

Then $[0,1] \subset \cup_{k=1}^{n-1} B_\varepsilon(\frac{k}{n})$.



Totally Bounded Sets

Example: Consider $X = [0, 1]$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

X is not totally bounded. To see this, take $\varepsilon = \frac{1}{2}$. Then for any x, $B_\varepsilon(x) = \{x\}$, so given any finite set $x_1, \ldots, x_n$,

$$\cup_{i=1}^{n} B_\varepsilon(x_i) = \{x_1, \ldots, x_n\} \not\supseteq [0, 1]$$

$B_{1/2}(x_i) = \{x_i\}$

However, X is bounded because $X = B_2(0)$.

bounded $\not\Rightarrow$ totally bounded

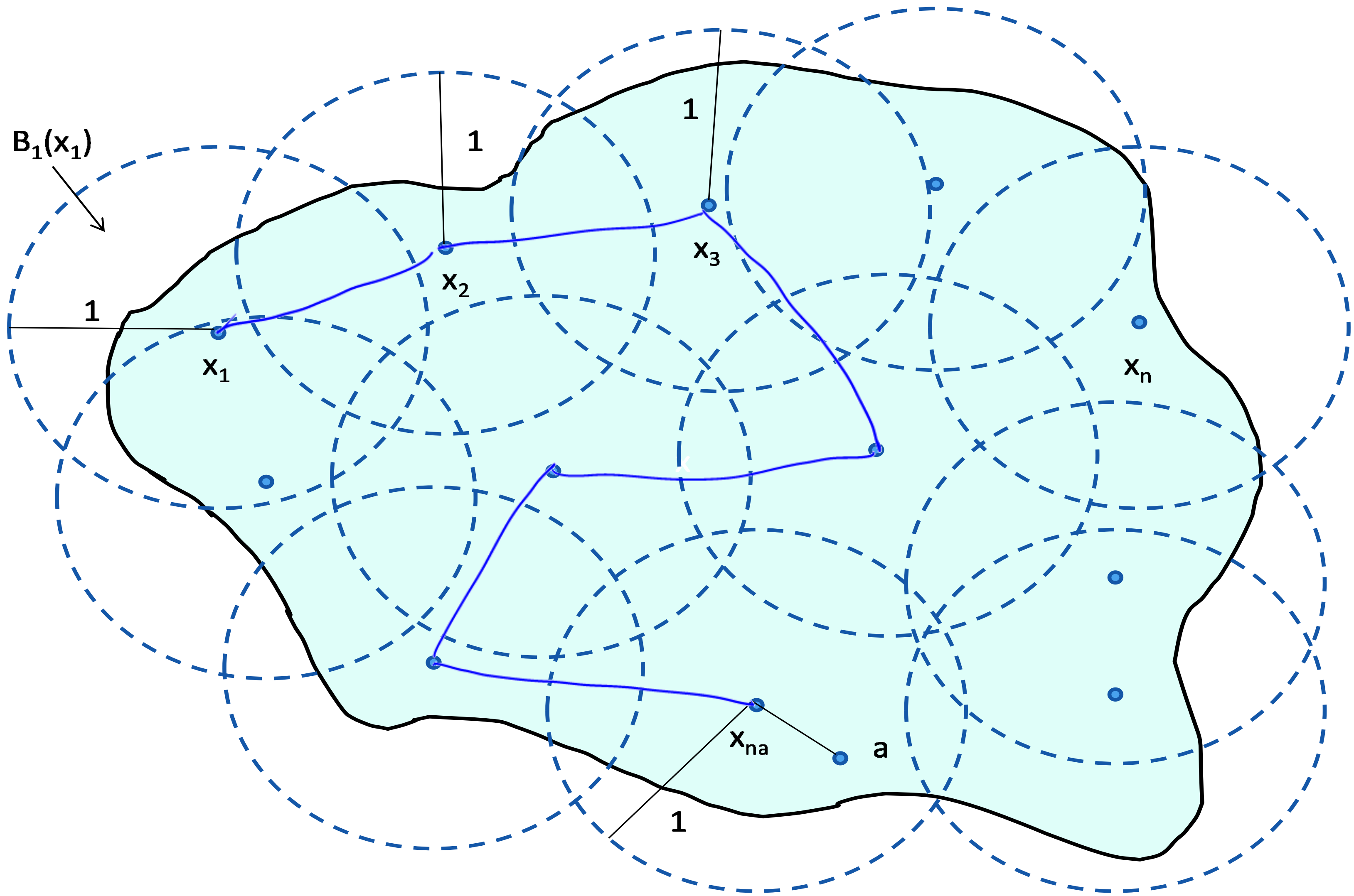
Totally Bounded Sets

Note that any totally bounded set in a metric space (X, d) is also bounded. To see this, let $A \subset X$ be totally bounded. Then $\exists x_1, \ldots, x_n \in A$ such that $A \subset B_1(x_1) \cup \cdots \cup B_1(x_n)$. Let

$$M = 1 + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)$$

Then $M < \infty$. Now fix $a \in A$. We claim $d(a, x_1) < M$. To see this, notice that there is some $n_a \in \{1, \ldots, n\}$ for which $a \in B_1(x_{n_a})$. Then

$$\begin{aligned} d(a, x_1) &\leq d(a, x_{n_a}) + \sum_{k=1}^{n-1} d(x_k, x_{k+1}) \\ &< 1 + \sum_{k=1}^{n-1} d(x_k, x_{k+1}) \\ &= M \end{aligned}$$



$B_1(x_1)$
1
x_1
1
x_2
1
x_3
x_n
x_{na}
a
1

Totally Bounded Sets

Remark 4. Every compact subset of a metric space is totally bounded:

$(\varepsilon > 0)$

Fix ε and consider the open cover

$$\mathcal{U}_\varepsilon = \{B_\varepsilon(a) : a \in A\}$$

If A is compact, then every open cover of A has a finite subcover; in particular, $\mathcal{U}_\varepsilon$ must have a finite subcover, but this just says that A is totally bounded.

$\Rightarrow \exists\ a_1, \ldots, a_n \in A$ s.t.

$$A \subseteq B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_n)$$

converse false: e.g. $(0,1]$ totally bounded but <u>not</u> compact

Compactness and Totally Bounded Sets

Theorem 5 (Thm. 8.16). *Let A be a subset of a metric space (X, d). Then A is compact if and only if it is complete and totally bounded.*

$\Rightarrow$: *Proof.* Here is a sketch of the proof; see de la Fuente for details. Compact implies totally bounded (Remark 4). Suppose $\{x_n\}$ is a Cauchy sequence in A. Since A is compact, A is sequentially compact, hence $\{x_n\}$ has a convergent subsequence $x_{n_k} \to a \in A$. Since $\{x_n\}$ is Cauchy, $x_n \to a$ (why?), so A is complete.

$\Leftarrow$: Conversely, suppose A is complete and totally bounded. Let $\{x_n\}$ be a sequence in A. Because A is totally bounded, we can extract a Cauchy subsequence $\{x_{n_k}\}$ (why?). Because A is complete, $x_{n_k} \to a$ for some $a \in A$, which shows that A is sequentially compact and hence compact. $\square$

Compact $\iff$ Closed and Totally Bounded

Putting these together: with results from lecture 5:

Corollary 1. *Let A be a subset of a complete metric space (X, d). Then A is compact if and only if A is closed and totally bounded.*

(X, d) complete, $A \subseteq X$ then

$$\begin{aligned} A \text{ compact} &\Rightarrow A \text{ complete and totally bounded} \\ &\Rightarrow A \text{ closed and totally bounded} \\ A \text{ closed and totally bounded} &\Rightarrow A \text{ complete and totally bounded} \\ &\Rightarrow A \text{ compact} \end{aligned}$$

Example: $[0,1]$ is compact in $\mathbf{E}^1$. ($\mathbb{R}$ with standard metric)

$\mathbb{E}^1$ complete, $[0,1]$ is closed and totally bounded
$\Rightarrow [0,1]$ is compact

Note: compact $\Rightarrow$ closed and bounded, but converse need not be true.

E.g. $[0,1]$ with the discrete metric.

$[0,1]$ with discrete metric is closed and bounded
but <u>not</u> totally bounded, so <u>not</u> compact

Heine-Borel Theorem - $\mathbf{E}^1$,

$\mathbb{R}$ with standard metric

Theorem 6 (Thm. 8.19, Heine-Borel)**.** *If $A \subseteq \mathbf{E}^1$, then A is compact if and only if A is closed and bounded.*

$\Leftarrow$: *Proof.* Let A be a closed, bounded subset of $\mathbf{R}$. Then $A \subseteq [a, b]$ for some interval $[a, b]$. Let $\{x_n\}$ be a sequence of elements of $[a, b]$. By the Bolzano-Weierstrass Theorem, $\{x_n\}$ contains a convergent subsequence with limit $x \in \mathbf{R}$. Since $[a, b]$ is closed, $x \in [a, b]$. Thus, we have shown that $[a, b]$ is sequentially compact, hence compact. A is a closed subset of $[a, b]$, hence A is compact.

$\Rightarrow$: Conversely, if A is compact, A is closed and bounded. $\square$

Heine-Borel Theorem - $\mathbf{E}^n$

Theorem 7 (Thm. 8.20, Heine-Borel)**.** *If $A \subseteq \mathbf{E}^n$, then A is compact if and only if A is closed and bounded.*

Proof. See de la Fuente. $\square$

Example: The closed interval

$$[a, b] = \{x \in \mathbf{R}^n : a_i \leq x_i \leq b_i \text{ for each } i = 1, \ldots, n\}$$

is compact in $\mathbf{E}^n$ for any $a, b \in \mathbf{R}^n$.

Continuous Images of Compact Sets

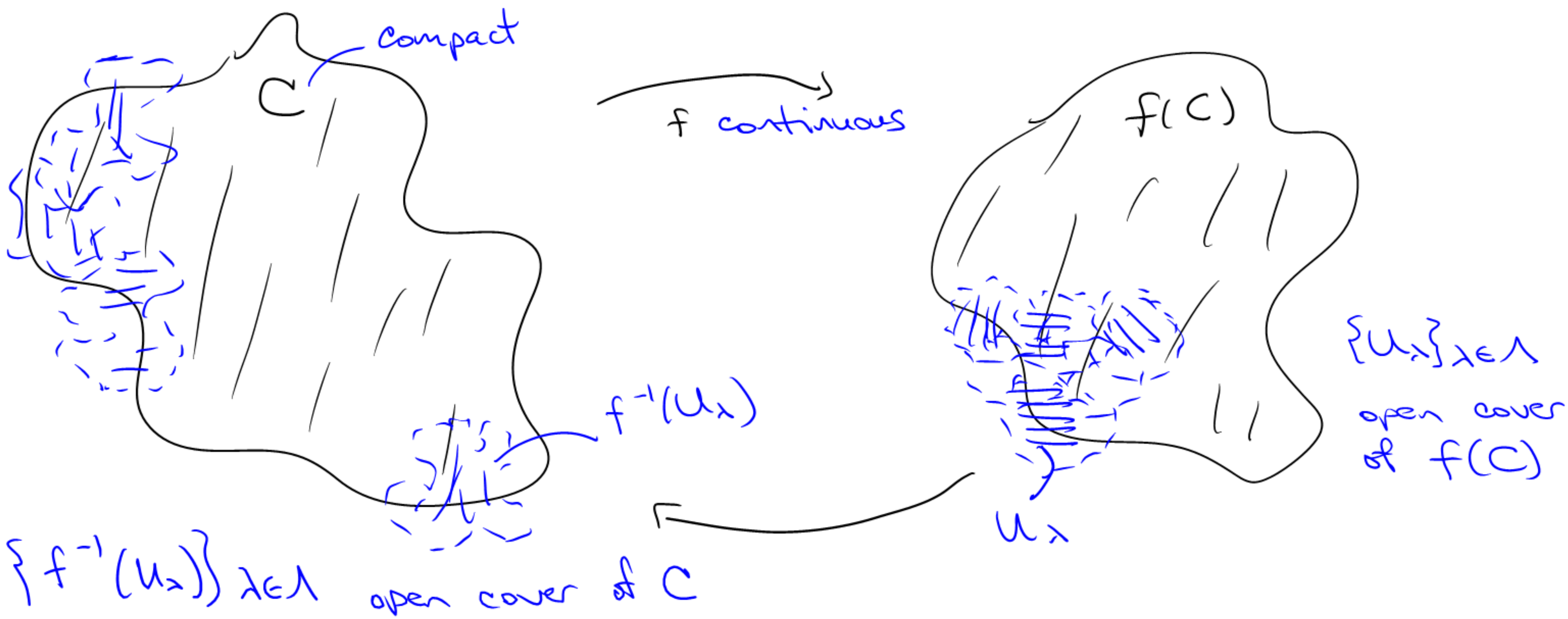
Theorem 8 (8.21). *Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact subset of (X, d), then $f(C)$ is compact in (Y, ρ).*

Proof. There is a proof in de la Fuente using sequential compactness. Here we give an alternative proof using directly the open cover definition of compactness.

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $f(C)$. For each point $c \in C$, $f(c) \in f(C)$ so $f(c) \in U_{\lambda_c}$ for some $\lambda_c \in \Lambda$, that is, $c \in f^{-1}\left(U_{\lambda_c}\right)$. Thus the collection $\left\{f^{-1}(U_\lambda) : \lambda \in \Lambda\right\}$ is a cover of C; in addition, since f is continuous, each set $f^{-1}(U_\lambda)$ is open in C, so $\left\{f^{-1}(U_\lambda) : \lambda \in \Lambda\right\}$ is an open cover of C. Since C is compact, there is a finite subcover

$$\left\{f^{-1}\left(U_{\lambda_1}\right), \ldots, f^{-1}\left(U_{\lambda_n}\right)\right\}$$

of C. Given $x \in f(C)$, there exists $c \in C$ such that $f(c) = x$, and $c \in f^{-1}\left(U_{\lambda_i}\right)$ for some i, so $\overset{=f(c)}{x} \in U_{\lambda_i}$. Thus, $\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$ is a finite subcover of $f(C)$, so $f(C)$ is compact. $\square$



Extreme Value Theorem

Corollary 2 (Thm. 8.22, Extreme Value Theorem)**.** *Let C be a compact set in a metric space (X, d), and suppose $f : C \to \mathbf{R}$ is continuous. Then f is bounded on C and attains its minimum and maximum on C.*

Proof. $f(C)$ $\subseteq \mathbb{R}$ is compact by Theorem 8.21, hence closed and bounded. Let $M = \sup f(C)$; $M < \infty$. Then $\forall m > 0$ there exists $y_m \in f(C)$ such that

$$M - \frac{1}{m} \leq y_m \leq M$$

So $y_m \to M$ and $\{y_m\} \subseteq f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e. there exists $c \in C$ such that $f(c) = M = \sup f(C)$, so f attains its maximum at c. The proof for the minimum is similar. $\square$

Compactness and Uniform Continuity

Theorem 9 (Thm. 8.24). *Let* (X, d) *and* (Y, ρ) *be metric spaces,* C *a compact subset of* X*, and* $f : C \to Y$ *continuous. Then* f *is uniformly continuous on* C*.*

Proof. Fix $\varepsilon > 0$. We ignore X and consider f as defined on the metric space (C, d). Given $c \in C$, find $\delta(c) > 0$ such that

$$x \in C,\ d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}$$

Let

$$U_c = B_{\delta(c)}(c)$$

Then

$$\{U_c : c \in C\}$$

is an open cover of C. Since C is compact, there is a finite subcover

$$\{U_{c_1}, \ldots, U_{c_n}\} \qquad C \subseteq \bigcup_{i=1}^{n} U_{c_i}$$

Let

$$\delta = \min\{\delta(c_1), \ldots, \delta(c_n)\} > 0$$

Given $x, y \in C$ with $d(x, y) < \delta$, note that $x \in U_{c_i}$ for some $i \in \{1, \ldots, n\}$, so $d(x, c_i) < \delta(c_i)$.

$$\begin{aligned} d(y, c_i) &\leq d(y, x) + d(x, c_i) \\ &< \delta + \delta(c_i) \\ &\leq \delta(c_i) + \delta(c_i) \\ &= 2\delta(c_i) \end{aligned}$$

so

$$
\begin{aligned}
\rho(f(x), f(y)) &\leq \rho(f(x), f(c_i)) + \rho(f(c_i), f(y)) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

which proves that f is uniformly continuous. $\square$