

Economics 204 Summer/Fall 2023
Lecture 10–Friday August 4, 2023

Diagonalization of Symmetric Real Matrices (from Handout)

Definition 1 Let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A basis $V = \{v_1, \dots, v_n\}$ of \mathbf{R}^n is *orthonormal* if $v_i \cdot v_j = \delta_{ij}$.

In other words, a basis is orthonormal if each basis element has unit length ($\|v_i\|^2 = v_i \cdot v_i = 1$ for each i), and distinct basis elements are perpendicular ($v_i \cdot v_j = 0$ for $i \neq j$).

Remark: Suppose that $x = \sum_{j=1}^n \alpha_j v_j$ where $\{v_1, \dots, v_n\}$ is an orthonormal basis of \mathbf{R}^n . Then

$$\begin{aligned} x \cdot v_k &= \left(\sum_{j=1}^n \alpha_j v_j \right) \cdot v_k \\ &= \sum_{j=1}^n \alpha_j (v_j \cdot v_k) \\ &= \sum_{j=1}^n \alpha_j \delta_{jk} \\ &= \alpha_k \end{aligned}$$

so

$$x = \sum_{j=1}^n (x \cdot v_j) v_j$$

Example: The standard basis of \mathbf{R}^n is orthonormal.

Recall that for a real $n \times m$ matrix A , A^\top denotes the transpose of A : the $(i, j)^{\text{th}}$ entry of A^\top is the $(j, i)^{\text{th}}$ entry of A . So the i^{th} row of A^\top is the i^{th} column of A .

Definition 2 A real $n \times n$ matrix A is *unitary* if $A^\top = A^{-1}$.

Theorem 3 A real $n \times n$ matrix A is unitary if and only if the columns of A are orthonormal.

Proof: Let v_j denote the j^{th} column of A .

$$\begin{aligned} A^\top = A^{-1} &\iff A^\top A = I \\ &\iff v_i \cdot v_j = \delta_{ij} \quad \forall i, j \\ &\iff \{v_1, \dots, v_n\} \text{ is orthonormal} \end{aligned}$$

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If A is unitary, let V be the set of columns of A and W be the standard basis of \mathbf{R}^n . Since A is unitary, it is invertible, so V is a basis of \mathbf{R}^n .

$$A^\top = A^{-1} = \text{Mtx}_{V,W}(\text{id})$$

Since V is orthonormal, the transformation between bases W and V preserves all geometry, including lengths and angles.

Theorem 4 *Let $T \in L(\mathbf{R}^n, \mathbf{R}^n)$ and W be the standard basis of \mathbf{R}^n . Suppose that $\text{Mtx}_W(T)$ is symmetric. Then the eigenvectors of T are all real, and there is an orthonormal basis $V = \{v_1, \dots, v_n\}$ of \mathbf{R}^n consisting of eigenvectors of T , so that $\text{Mtx}_W(T)$ is diagonalizable:*

$$\text{Mtx}_W(T) = \text{Mtx}_{W,V}(\text{id}) \cdot \text{Mtx}_V(T) \cdot \text{Mtx}_{V,W}(\text{id})$$

where $\text{Mtx}_V T$ is diagonal and the change of basis matrices $\text{Mtx}_{V,W}(\text{id})$ and $\text{Mtx}_{W,V}(\text{id})$ are unitary.

Proof: (*Sketch*) The proof of the theorem requires a lengthy digression into the linear algebra of complex vector spaces. Here is a very brief outline.

1. Let $M = \text{Mtx}_W(T)$.
2. The inner product in \mathbf{C}^n is defined as follows:

$$x \cdot y = \sum_{j=1}^n x_j \cdot \bar{y}_j$$

where \bar{c} denotes the complex conjugate of any $c \in \mathbf{C}$; note that this implies that $x \cdot y = \bar{y} \cdot \bar{x}$. The usual inner product in \mathbf{R}^n is the restriction of this inner product on \mathbf{C}^n to \mathbf{R}^n .

3. Given any complex matrix A , define A^* to be the matrix whose $(i, j)^{\text{th}}$ entry is \bar{a}_{ji} ; in other words, A^* is formed by taking the complex conjugate of each element of the transpose of A . It is easy to verify that given $x, y \in \mathbf{C}^n$ and a complex $n \times n$ matrix A , $Ax \cdot y = x \cdot A^*y$. Since M is real and symmetric, $M^* = M$.
4. If M is real and symmetric, and $\lambda \in \mathbf{C}$ is an eigenvalue of M , with eigenvector $x \in \mathbf{C}^n$, then

$$\begin{aligned} \lambda|x|^2 &= \lambda(x \cdot x) \\ &= (\lambda x) \cdot x \\ &= (Mx) \cdot x \\ &= x \cdot (M^*x) \end{aligned}$$

$$\begin{aligned}
&= x \cdot (Mx) \\
&= x \cdot (\lambda x) \\
&= \frac{x \cdot (\lambda x)}{(\lambda x) \cdot x} \\
&= \frac{x \cdot (\lambda x)}{\lambda(x \cdot x)} \\
&= \frac{\lambda(x \cdot x)}{\lambda(x \cdot x)} \\
&= \bar{\lambda} |x|^2
\end{aligned}$$

which proves that $\lambda = \bar{\lambda}$, hence $\lambda \in \mathbf{R}$.

5. If M is real (not necessarily symmetric) and $\lambda \in \mathbf{R}$ is an eigenvalue, then $\det(M - \lambda I) = 0 \Rightarrow \exists v \in \mathbf{R}^n$ s.t. $(M - \lambda I)v = 0$, so there is at least one real eigenvector. Symmetry implies that, if λ has multiplicity m , there are m independent real eigenvectors corresponding to λ (but unfortunately we don't have time to show this). Thus, there is a basis of eigenvectors, hence M is diagonalizable over \mathbf{R} .
6. If M is real and symmetric, eigenvectors corresponding to distinct eigenvalues are orthogonal: Suppose that $Mx = \lambda x$ and $My = \rho y$ with $\rho \neq \lambda$. Then

$$\begin{aligned}
\lambda(x \cdot y) &= (\lambda x) \cdot y \\
&= (Mx) \cdot y \\
&= (Mx)^\top y \\
&= (x^\top M^\top) y \\
&= (x^\top M) y \\
&= x^\top (My) \\
&= x^\top (\rho y) \\
&= x \cdot (\rho y) \\
&= \rho(x \cdot y)
\end{aligned}$$

so $(\lambda - \rho)(x \cdot y) = 0$; since $\lambda - \rho \neq 0$, we must have $x \cdot y = 0$.

7. Using the Gram-Schmidt method, we can get an orthonormal basis of eigenvectors:

- Let $X_\lambda = \{x \in \mathbf{R}^n : Mx = \lambda x\}$, the set of all eigenvectors corresponding to λ . Notice that if $Mx = \lambda x$ and $My = \lambda y$, then

$$M(\alpha x + \beta y) = \alpha Mx + \beta My = \alpha \lambda x + \beta \lambda y = \lambda(\alpha x + \beta y)$$

so X_λ is a vector subspace. Thus, given any basis of X_λ , we wish to find an orthonormal basis of X_λ ; all elements of this orthonormal basis will be eigenvectors corresponding to λ .

- Suppose X_λ is m -dimensional and we are given independent vectors $x_1, \dots, x_m \in X_\lambda$. The Gram-Schmidt method finds an orthonormal basis $\{v_1, \dots, v_m\}$ for X_λ .
- Let $v_1 = \frac{x_1}{|x_1|}$. Note that $|v_1| = 1$.

- Suppose we have found an orthonormal set $\{v_1, \dots, v_k\}$ such that $\text{span}\{v_1, \dots, v_k\} = \text{span}\{x_1, \dots, x_k\}$, with $k < m$. Let

$$y_{k+1} = x_{k+1} - \sum_{j=1}^k (x_{k+1} \cdot v_j) v_j, \quad v_{k+1} = \frac{y_{k+1}}{|y_{k+1}|}$$

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$$\begin{aligned} \text{span}\{v_1, \dots, v_{k+1}\} &= \text{span}\{v_1, \dots, v_k, v_{k+1}\} \\ &= \text{span}\{v_1, \dots, v_k, y_{k+1}\} \\ &= \text{span}\{v_1, \dots, v_k, x_{k+1}\} \\ &= \text{span}\{x_1, \dots, x_k, x_{k+1}\} \end{aligned}$$

- For $i = 1, \dots, k$,

$$\begin{aligned} y_{k+1} \cdot v_i &= \left(x_{k+1} - \sum_{j=1}^k (x_{k+1} \cdot v_j) v_j \right) \cdot v_i \\ &= x_{k+1} \cdot v_i - \sum_{j=1}^k (x_{k+1} \cdot v_j) (v_j \cdot v_i) \\ &= x_{k+1} \cdot v_i - \sum_{j=1}^k (x_{k+1} \cdot v_j) \delta_{ij} \\ &= x_{k+1} \cdot v_i - x_{k+1} \cdot v_i \\ &= 0 \\ v_{k+1} \cdot v_i &= \frac{y_{k+1} \cdot v_i}{|y_{k+1}|} \\ &= \frac{0}{|y_{k+1}|} \\ &= 0 \\ |v_{k+1}| &= \frac{|y_{k+1}|}{|y_{k+1}|} \\ &= 1 \end{aligned}$$

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Application to Quadratic Forms

Consider a quadratic form

$$f(x_1, \dots, x_n) = \sum_{i=1}^n \alpha_{ii} x_i^2 + \sum_{i < j} \beta_{ij} x_i x_j \quad (1)$$

Let

$$\alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ji}}{2} & \text{if } i > j \end{cases}$$

Let

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}$$

so

$$f(x) = x^\top Ax$$

Example: Let

$$f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

Let

$$A = \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix}$$

so A is symmetric and

$$\begin{aligned} (x_1, x_2) \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= (x_1, x_2) \begin{pmatrix} \alpha x_1 + \frac{\beta}{2} x_2 \\ \frac{\beta}{2} x_1 + \gamma x_2 \end{pmatrix} \\ &= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 \\ &= f(x) \end{aligned}$$

Returning to the general quadratic form in Equation (1), A is symmetric, so let $V = \{v_1, \dots, v_n\}$ be an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Then

$$\begin{aligned} A &= U^\top D U \\ \text{where } D &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \\ \text{and } U &= \text{Mtx}_{V,W}(id) \text{ is unitary} \end{aligned}$$

The columns of U^\top (the rows of U) are the coordinates of v_1, \dots, v_n , expressed in terms of the standard basis W .

Given $x \in \mathbf{R}^n$, recall

$$x = \sum_{i=1}^n \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$

Then

$$f(x) = f\left(\sum \gamma_i v_i\right)$$

$$\begin{aligned}
&= \left(\sum \gamma_i v_i\right)^\top A \left(\sum \gamma_i v_i\right) \\
&= \left(\sum \gamma_i v_i\right)^\top U^\top D U \left(\sum \gamma_i v_i\right) \\
&= \left(U \sum \gamma_i v_i\right)^\top D \left(U \sum \gamma_i v_i\right) \\
&= \left(\sum \gamma_i U v_i\right)^\top D \left(\sum \gamma_i U v_i\right) \\
&= (\gamma_1, \dots, \gamma_n) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \\
&= \sum \lambda_i \gamma_i^2
\end{aligned}$$

The equation for the level sets of f is

$$\sum_{i=1}^n \lambda_i \gamma_i^2 = C$$

- If $\lambda_i \geq 0$ for all i , the level set is an ellipsoid, with principal axes in the directions v_1, \dots, v_n . The length of the principal axis along v_i is $\sqrt{C/\lambda_i}$ if $C \geq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C < 0$. See Figure 1.
- If $\lambda_i \leq 0$ for all i , the level set is an ellipsoid, with principal axes in the directions v_1, \dots, v_n . The length of the principal axis along v_i is $\sqrt{C/\lambda_i}$ if $C \leq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C > 0$.
- If $\lambda_i > 0$ for some i and $\lambda_j < 0$ for some j , the level set is a hyperboloid. For example, suppose $n = 2$, $\lambda_1 > 0$, $\lambda_2 < 0$. The equation is

$$\begin{aligned}
C &= \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2 \\
&= \left(\sqrt{\lambda_1} \gamma_1 + \sqrt{|\lambda_2|} \gamma_2\right) \left(\sqrt{\lambda_1} \gamma_1 - \sqrt{|\lambda_2|} \gamma_2\right)
\end{aligned}$$

This is a hyperbola with asymptotes

$$\begin{aligned}
0 &= \sqrt{\lambda_1} \gamma_1 + \sqrt{|\lambda_2|} \gamma_2 \\
\Rightarrow \gamma_1 &= -\sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2 \\
0 &= \left(\sqrt{\lambda_1} \gamma_1 - \sqrt{|\lambda_2|} \gamma_2\right) \\
\Rightarrow \gamma_1 &= \sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2
\end{aligned}$$

See Figure 2. This proves the following corollary of Theorem 4.

Corollary 5 Consider the quadratic form (1).

1. f has a global minimum at 0 if and only if $\lambda_i \geq 0$ for all i ; the level sets of f are ellipsoids with principal axes aligned with the orthonormal eigenvectors v_1, \dots, v_n .
2. f has a global maximum at 0 if and only if $\lambda_i \leq 0$ for all i ; the level sets of f are ellipsoids with principal axes aligned with the orthonormal eigenvectors v_1, \dots, v_n .
3. If $\lambda_i < 0$ for some i and $\lambda_j > 0$ for some j , then f has a saddle point at 0; the level sets of f are hyperboloids with principal axes aligned with the orthonormal eigenvectors v_1, \dots, v_n .

Section 3.4. Linear Maps between Normed Spaces

Definition 6 Suppose X, Y are normed vector spaces and $T \in L(X, Y)$. We say T is bounded if

$$\exists \beta \in \mathbf{R} \text{ s.t. } \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X$$

Note this implies that T is Lipschitz with constant β .

Theorem 7 (Thms. 4.1, 4.3) Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then

$$\begin{aligned} & T \text{ is continuous at some point } x_0 \in X \\ \iff & T \text{ is continuous at every } x \in X \\ \iff & T \text{ is uniformly continuous on } X \\ \iff & T \text{ is Lipschitz} \\ \iff & T \text{ is bounded} \end{aligned}$$

Proof: Suppose T is continuous at x_0 . Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\|z - x_0\| < \delta \Rightarrow \|T(z) - T(x_0)\| < \varepsilon$$

Now suppose x is any element of X . If $\|y - x\| < \delta$, let $z = y - x + x_0$, so $\|z - x_0\| = \|y - x\| < \delta$.

$$\begin{aligned} & \|T(y) - T(x)\| \\ &= \|T(y - x)\| \\ &= \|T(y - x + x_0 - x_0)\| \\ &= \|T(z) - T(x_0)\| \\ &< \varepsilon \end{aligned}$$

which proves that T is continuous at every x , and uniformly continuous.

We claim that T is bounded if and only if T is continuous at 0. Suppose T is not bounded. Then

$$\exists \{x_n\} \text{ s.t. } \|T(x_n)\| > n\|x_n\| \quad \forall n$$

Note that $x_n \neq 0$. Let $\varepsilon = 1$. Fix $\delta > 0$ and choose n such that $\frac{1}{n} < \delta$. Let

$$\begin{aligned} x'_n &= \frac{x_n}{n\|x_n\|} \\ \|x'_n\| &= \frac{\|x_n\|}{n\|x_n\|} \\ &= \frac{1}{n} \\ &< \delta \\ \|T(x'_n) - T(0)\| &= \|T(x'_n)\| \\ &= \frac{1}{n\|x_n\|} \|T(x_n)\| \\ &> \frac{n\|x_n\|}{n\|x_n\|} \\ &= 1 \\ &= \varepsilon \end{aligned}$$

Since this is true for every δ , T is not continuous at 0. Therefore, T continuous at 0 implies T is bounded. Now, suppose T is bounded, so find M such that $\|T(x)\| \leq M\|x\|$ for every $x \in X$. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then

$$\begin{aligned} \|x - 0\| < \delta &\Rightarrow \|x\| < \delta \\ &\Rightarrow \|T(x) - T(0)\| = \|T(x)\| < M\delta \\ &\Rightarrow \|T(x) - T(0)\| < \varepsilon \end{aligned}$$

so T is continuous at 0.

Thus, we have shown that continuity at some point x_0 implies uniform continuity, which implies continuity at every point, which implies T is continuous at 0, which implies that T is bounded, which implies that T is continuous at 0, which implies that T is continuous at some x_0 , so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose T is bounded, with constant M . Then

$$\begin{aligned} \|T(x) - T(y)\| &= \|T(x - y)\| \\ &\leq M\|x - y\| \end{aligned}$$

so T is Lipschitz with constant M ; conversely, if T is Lipschitz with constant M , then T is bounded with constant M . So all the statements are equivalent. ■

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

Theorem 8 (Thm. 4.5) *Let X, Y be normed vector spaces with $\dim X = n$. Every $T \in L(X, Y)$ is bounded.*

Proof: See de la Fuente. ■

Definition 9 A *topological isomorphism* between normed vector spaces X and Y is a linear transformation $T \in L(X, Y)$ that is invertible (one-to-one, onto), continuous, and has a continuous inverse.

Two normed vector spaces X and Y are *topologically isomorphic* if there is a topological isomorphism $T : X \rightarrow Y$.

Suppose X and Y are normed vector spaces. We define

$$\begin{aligned} B(X, Y) &= \{T \in L(X, Y) : T \text{ is bounded}\} \\ \|T\|_{B(X, Y)} &= \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X}, x \in X, x \neq 0 \right\} \\ &= \sup \{\|T(x)\|_Y : \|x\|_X = 1\} \end{aligned}$$

Theorem 10 (Thm. 4.8) *Let X, Y be normed vector spaces. Then*

$$(B(X, Y), \|\cdot\|_{B(X, Y)})$$

is a normed vector space.

Proof: See de la Fuente. ■

Theorem 11 (Thm. 4.9) *Let $T \in L(\mathbf{R}^n, \mathbf{R}^m)$ ($= B(\mathbf{R}^n, \mathbf{R}^m)$) with matrix $A = (a_{ij})$ with respect to the standard bases. Let*

$$M = \max\{|a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then

$$M \leq \|T\| \leq M\sqrt{mn}$$

Proof: See de la Fuente. ■

Theorem 12 (Thm. 4.10) *Let $R \in L(\mathbf{R}^m, \mathbf{R}^n)$ and $S \in L(\mathbf{R}^n, \mathbf{R}^p)$. Then*

$$\|S \circ R\| \leq \|S\| \|R\|$$

Proof: See de la Fuente. ■

Define

$$\Omega(\mathbf{R}^n) = \{T \in L(\mathbf{R}^n, \mathbf{R}^n) : T \text{ is invertible}\}$$

Theorem 13 (Thm. 4.11') *Suppose $T \in L(\mathbf{R}^n, \mathbf{R}^n)$ and E is the standard basis of \mathbf{R}^n . Then*

T is invertible

$$\Leftrightarrow \ker T = \{0\}$$

$$\Leftrightarrow \det(Mtx_E(T)) \neq 0$$

$$\Leftrightarrow \det(Mtx_{V,V}(T)) \neq 0 \text{ for every basis } V$$

$$\Leftrightarrow \det(Mtx_{V,W}(T)) \neq 0 \text{ for every pair of bases } V, W$$

Theorem 14 (Thm. 4.12) *If $S, T \in \Omega(\mathbf{R}^n)$, then $S \circ T \in \Omega(\mathbf{R}^n)$ and*

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}$$

Theorem 15 (Thm. 4.14) *Let $S, T \in L(\mathbf{R}^n, \mathbf{R}^n)$. If T is invertible and*

$$\|T - S\| < \frac{1}{\|T^{-1}\|}$$

then S is invertible. In particular, $\Omega(\mathbf{R}^n)$ is open in $L(\mathbf{R}^n, \mathbf{R}^n) = B(\mathbf{R}^n, \mathbf{R}^n)$.

Proof: See de la Fuente. ■

Theorem 16 (4.15) *The function $(\cdot)^{-1} : \Omega(\mathbf{R}^n) \rightarrow \Omega(\mathbf{R}^n)$ that assigns T^{-1} to each $T \in \Omega(\mathbf{R}^n)$ is continuous.*

Proof: See de la Fuente. ■

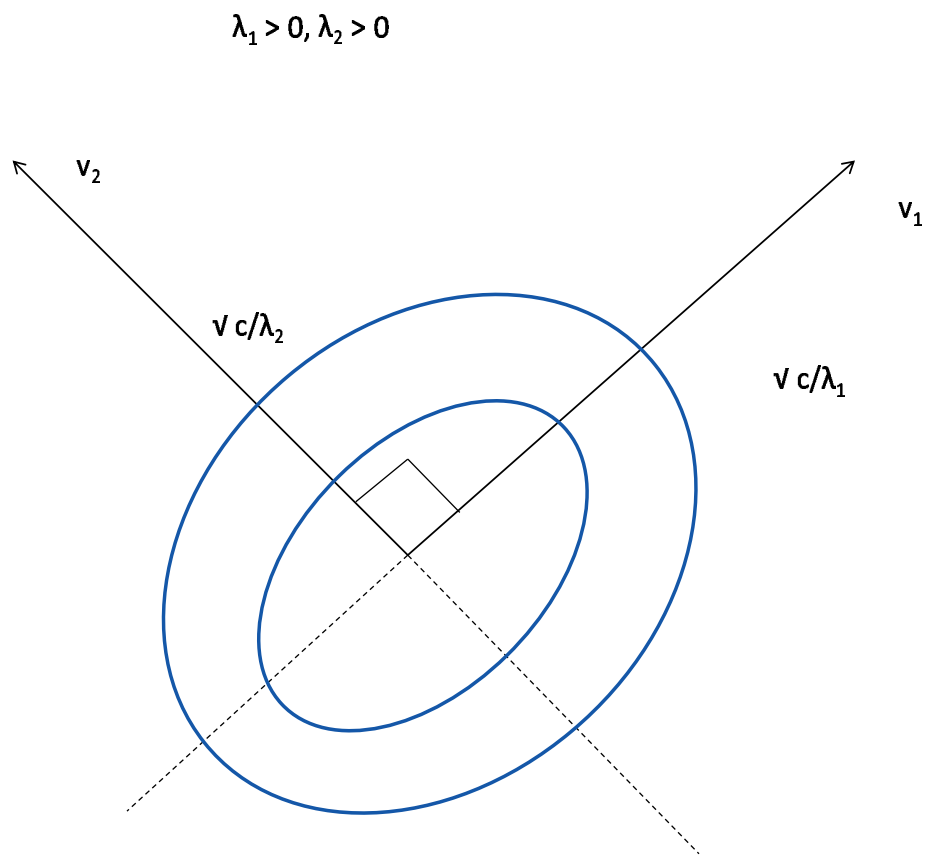


Figure 1: If $\lambda_1, \lambda_2 > 0$ and $C > 0$, the level set is an ellipsoid, with principal axes in the directions v_1, v_2 . The length of the principal axis along v_i is $\sqrt{C/\lambda_i}$.

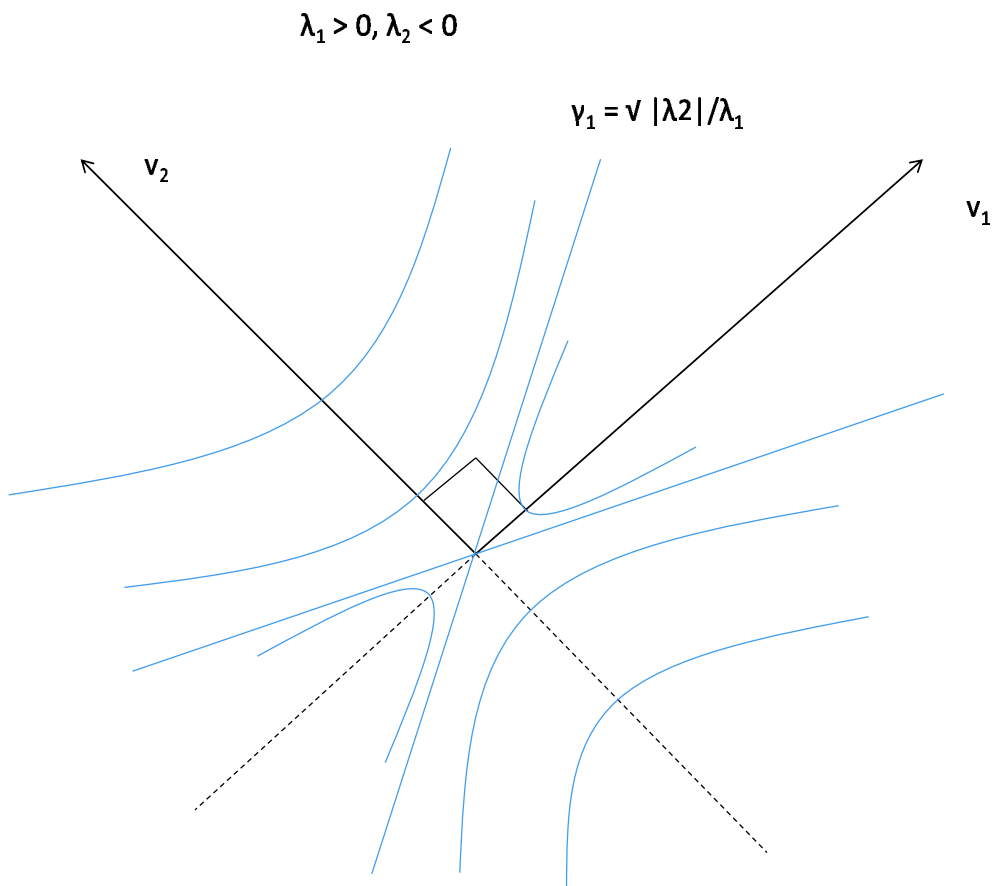


Figure 2: If $\lambda_1 > 0$ and $\lambda_2 < 0$, the level set is a hyperbola with asymptotes $\gamma_1 = \sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2$.