Economics 204 Summer/Fall 2023
Lecture 11–Monday August 7, 2023

Sections 4.1-4.3 (Unified)

Definition 1 Let $f : I \to \mathbf{R}$, where $I \subseteq \mathbf{R}$ is an open interval. f is *differentiable* at $x \in I$ if

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = a$$

for some $a \in \mathbf{R}$.

This is equivalent to $\exists a \in \mathbf{R}$ such that:

$$\begin{aligned}
& \lim_{h\to 0} \frac{f(x+h) - (f(x) + ah)}{h} = 0 \\
\Leftrightarrow \quad & \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(x+h) - (f(x) + ah)}{h} \right| < \varepsilon \\
\Leftrightarrow \quad & \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \frac{|f(x+h) - (f(x) + ah)|}{|h|} < \varepsilon \\
\Leftrightarrow \quad & \lim_{h\to 0} \frac{|f(x+h) - (f(x) + ah)|}{|h|} = 0
\end{aligned}$$

Recall that the limit considers h near zero, but not $h = 0$.

Definition 2 If $X \subseteq \mathbf{R}^n$ is open, $f : X \to \mathbf{R}^m$ is *differentiable at* $x \in X$ if[1]

$$\exists T_x \in L(\mathbf{R}^n, \mathbf{R}^m) \text{ s.t. } \lim_{h\to 0, h\in\mathbf{R}^n} \frac{|f(x+h) - (f(x) + T_x(h))|}{|h|} = 0 \tag{1}$$

f is *differentiable* if it is differentiable at all $x \in X$.

Note that T_x is uniquely determined by Equation (1). h is a small, nonzero element of $\mathbf{R}^n$; $h \to 0$ from any direction, from above, below, along a spiral, etc. The definition requires that one linear operator T_x works no matter how h approaches zero. In this case, $f(x) + T_x(h)$ is the best linear approximation to $f(x+h)$ for small h.

Notation:

- $y = O(|h|^n)$ as $h \to 0$ – read "y is big-Oh of $|h|^n$" – means

$$\exists K, \delta > 0 \text{ s.t. } |h| < \delta \Rightarrow |y| \le K|h|^n$$

[1] Recall $|\cdot|$ denotes the Euclidean distance.

- $y = o(|h|^n)$ as $h \to 0$ – read "y is little-oh of $|h|^n$" – means

$$\lim_{h \to 0} \frac{|y|}{|h|^n} = 0$$

Note that the statement $y = O(|h|^{n+1})$ as $h \to 0$ implies $y = o(|h|^n)$ as $h \to 0$.

Also note that if y is either $O(|h|^n)$ or $o(|h|^n)$, then $y \to 0$ as $h \to 0$; the difference in whether y is "big-Oh" or "little-oh" tells us something about the *rate* at which $y \to 0$.

Using this notation, note that f is differentiable at $x \Leftrightarrow \exists T_x \in L(\mathbf{R}^n, \mathbf{R}^m)$ such that

$$f(x+h) = f(x) + T_x(h) + o(h) \text{ as } h \to 0$$

Notation:

- df_x is the linear transformation T_x
- $Df(x)$ is the matrix of df_x with respect to the standard basis.
  This is called the *Jacobian* or *Jacobian matrix* of f at x
- $E_f(h) = f(x+h) - (f(x) + df_x(h))$ is the *error term*

Using this notation,

$$f \text{ is differentiable at } x \Leftrightarrow E_f(h) = o(h) \text{ as } h \to 0$$

Now compute $Df(x) = (a_{ij})$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbf{R}^n$. Look in direction e_j (note that $|\gamma e_j| = |\gamma|$).

$$\begin{aligned}
o(\gamma) &= f(x+\gamma e_j) - (f(x) + T_x(\gamma e_j)) \\
&= f(x+\gamma e_j) - \left(f(x) + \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \\
&= f(x+\gamma e_j) - \left(f(x) + \begin{pmatrix} \gamma a_{1j} \\ \vdots \\ \gamma a_{mj} \end{pmatrix} \right)
\end{aligned}$$

For $i = 1, \dots, m$, let f^i denote the i^{th} component of the function f:

$$\begin{aligned} f^i(x + \gamma e_j) - \left(f^i(x) + \gamma a_{ij}\right) &= o(\gamma) \\ \text{so } a_{ij} &= \frac{\partial f^i}{\partial x_j}(x) \end{aligned}$$

Theorem 3 (Thm. 3.3) *Suppose $X \subseteq \mathbf{R}^n$ is open and $f : X \to \mathbf{R}^m$ is differentiable at $x \in X$. Then $\frac{\partial f^i}{\partial x_j}$ exists at x for $1 \le i \le m$, $1 \le j \le n$, and*

$$Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_n}(x) \end{pmatrix}$$

i.e. the Jacobian at x is the matrix of partial derivatives at x.

Remark: If f is differentiable at x, then all first-order partial derivatives $\frac{\partial f^i}{\partial x_j}$ exist at x. However, the converse is false: existence of all the first-order partial derivatives does not imply that f is differentiable. The missing piece is continuity of the partial derivatives:

Theorem 4 (Thm. 3.4) *If all the first-order partial derivatives $\frac{\partial f^i}{\partial x_j}$ ($1 \le i \le m$, $1 \le j \le n$) exist and are continuous at x, then f is differentiable at x.*

Directional Derivatives:

Suppose $X \subseteq \mathbf{R}^n$ open, $f : X \to \mathbf{R}^m$ is differentiable at x, and $|u| = 1$.

$$\begin{aligned} & f(x + \gamma u) - (f(x) + T_x(\gamma u)) = o(\gamma) \text{ as } \gamma \to 0 \\ \Rightarrow \quad & f(x + \gamma u) - (f(x) + \gamma T_x(u)) = o(\gamma) \text{ as } \gamma \to 0 \\ \Rightarrow \quad & \lim_{\gamma \to 0} \frac{f(x + \gamma u) - f(x)}{\gamma} = T_x(u) = Df(x)u \end{aligned}$$

i.e. the directional derivative in the direction u (with $|u| = 1$) is

$$Df(x)u \in \mathbf{R}^m$$

Theorem 5 (Thm. 3.5, Chain Rule) *Let $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$ be open, $f : X \to Y$, $g : Y \to \mathbf{R}^p$. Let $x_0 \in X$ and $F = g \circ f$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $F = g \circ f$ is differentiable at x_0 and*

$$\begin{aligned} dF_{x_0} &= dg_{f(x_0)} \circ df_{x_0} \\ & \quad \textit{(composition of linear transformations)} \\ DF(x_0) &= Dg(f(x_0))Df(x_0) \\ & \quad \textit{(matrix multiplication)} \end{aligned}$$

Remark: The statement is exactly the same as in the univariate case, except we replace the univariate derivative by a linear transformation. The proof is more or less the same, with a bit of linear algebra added.

Theorem 6 (Thm. 1.7, Mean Value Theorem, Univariate Case) *Let $a, b \in \mathbf{R}$. Suppose $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

that is, such that

$$f(b) - f(a) = f'(c)(b - a)$$

Proof: Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then $g(a) = 0 = g(b)$. See Figure 1. Note that for $x \in (a, b)$,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so it suffices to find $c \in (a, b)$ such that $g'(c) = 0$.

Case I: If $g(x) = 0$ for all $x \in [a, b]$, choose an arbitrary $c \in (a, b)$, and note that $g'(c) = 0$, so we are done.

Case II: Suppose $g(x) > 0$ for some $x \in [a, b]$. Since g is continuous on $[a, b]$, it attains its maximum at some point $c \in (a, b)$. Since g is differentiable at c and c is an interior point of the domain of g, we have $g'(c) = 0$, and we are done.

Case III: If $g(x) < 0$ for some $x \in [a, b]$, the argument is similar to that in Case II. ■

Remark: The Mean Value Theorem is useful for estimating bounds on functions and error terms in approximation of functions.

Notation:

$$\ell(x, y) = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$$

is the line segment from x to y.

Theorem 7 (Mean Value Theorem) *Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable on an open set $X \subseteq \mathbf{R}^n$, $x, y \in X$ and $\ell(x, y) \subseteq X$. Then there exists $z \in \ell(x, y)$ such that*

$$f(y) - f(x) = Df(z)(y - x)$$

Remark: This statement is different from Theorem 3.7 in de la Fuente. Notice that the statement is exactly the same as in the univariate case. For $f : \mathbf{R}^n \to \mathbf{R}^m$, we can apply the Mean Value Theorem to each component, to obtain $z_1, \ldots, z_m \in \ell(x, y)$ such that

$$f^i(y) - f^i(x) = Df^i(z_i)(y - x)$$

However, we cannot find a single z which works for every component. Note that each $z_i \in \ell(x, y) \subset \mathbf{R}^n$; there are m of them, one for each component in the range.

The following result plays the same role in estimating function values and error terms for functions taking values in $\mathbf{R}^m$ as the Mean Value Theorem plays for functions from $\mathbf{R}$ to $\mathbf{R}$.

Theorem 8 *Suppose $X \subset \mathbf{R}^n$ is open and $f : X \to \mathbf{R}^m$ is differentiable. If $x, y \in X$ and $\ell(x, y) \subseteq X$, then there exists $z \in \ell(x, y)$ such that*

$$\begin{aligned} |f(y) - f(x)| &\leq |df_z(y - x)| \\ &\leq \|df_z\||y - x| \end{aligned}$$

Remark: To understand why we don't get equality, consider $f : [0, 1] \to \mathbf{R}^2$ defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

f maps $[0, 1]$ to the unit circle in $\mathbf{R}^2$. Note that $f(0) = f(1) = (1, 0)$, so $|f(1) - f(0)| = 0$. However, for any $z \in [0, 1]$,

$$\begin{aligned} |df_z(1 - 0)| &= |2\pi(-\sin 2\pi z, \cos 2\pi z)| \\ &= 2\pi\sqrt{\sin^2 2\pi z + \cos^2 2\pi z} \\ &= 2\pi \end{aligned}$$

Section 4.4. Taylor's Theorem

Theorem 9 (Thm. 1.9, Taylor's Theorem in $\mathbf{R}^1$) *Let $f : I \to \mathbf{R}$ be n-times differentiable, where $I \subseteq \mathbf{R}$ is an open interval. If $x, x + h \in I$, then*

$$f(x + h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)h^k}{k!} + E_n$$

where $f^{(k)}$ is the k^{th} derivative of f and

$$E_n = \frac{f^{(n)}(x + \lambda h)h^n}{n!} \text{ for some } \lambda \in (0, 1)$$

Motivation: Let

$$\begin{aligned}
T_n(h) &= f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} \\
&= f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \cdots + \frac{f^{(n)}(x)h^n}{n!} \\
T_n(0) &= f(x) \\
T_n'(h) &= f'(x) + f''(x)h + \cdots + \frac{f^{(n)}(x)h^{n-1}}{(n-1)!} \\
T_n'(0) &= f'(x) \\
T_n''(h) &= f''(x) + \cdots + \frac{f^{(n)}(x)h^{n-2}}{(n-2)!} \\
T_n''(0) &= f''(x) \\
&\vdots \\
T_n^{(n)}(0) &= f^{(n)}(x)
\end{aligned}$$

so $T_n(h)$ is the unique n^{th} degree polynomial such that

$$\begin{aligned}
T_n(0) &= f(x) \\
T_n'(0) &= f'(x) \\
&\vdots \\
T_n^{(n)}(0) &= f^{(n)}(x)
\end{aligned}$$

The proof of the formula for the remainder E_n is essentially the Mean Value Theorem; the problem in applying it is that the point $x + \lambda h$ is not known in advance.

Theorem 10 (Alternate Taylor's Theorem in $\mathbf{R}^1$) *Let $f : I \to \mathbf{R}$ be n times differentiable, where $I \subseteq \mathbf{R}$ is an open interval and $x \in I$. Then*

$$f(x+h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + o\left(h^n\right) \quad \text{as } h \to 0$$

If f is $(n+1)$ times continuously differentiable (i.e. all derivatives up to order $n+1$ exist and are continuous), then

$$f(x+h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + O\left(h^{n+1}\right) \quad \text{as } h \to 0$$

Remark: The first equation in the statement of the theorem is essentially a restatement of the definition of the n^{th} derivative. The second statement is proven from Theorem 1.9, and the continuity of the derivative, hence the boundedness of the derivative on a neighborhood of x.

Definition 11 Let $X \subseteq \mathbf{R}^n$ be open. A function $f : X \to \mathbf{R}^m$ is *continuously differentiable* on X if

- f is differentiable on X and
- df_x is a continuous function of x from X to $L(\mathbf{R}^n, \mathbf{R}^m)$, with operator norm $\|df_x\|$

f is C^k if all partial derivatives of order less than or equal to k exist and are continuous in X.

Theorem 12 (Thm. 4.3) *Suppose $X \subseteq \mathbf{R}^n$ is open and $f : X \to \mathbf{R}^m$. Then f is continuously differentiable on X if and only if f is C^1.*

Remark: The notation in Taylor's Theorem is difficult. If $f : \mathbf{R}^n \to \mathbf{R}^m$, the quadratic terms are not hard for $m = 1$; for $m > 1$, we handle each component separately. For cubic and higher order terms, the notation is a mess.

Linear Terms:

Theorem 13 *Suppose $X \subseteq \mathbf{R}^n$ is open and $x \in X$. If $f : X \to \mathbf{R}^m$ is differentiable, then*

$$f(x+h) = f(x) + Df(x)h + o(h) \text{ as } h \to 0$$

The previous theorem is essentially a restatement of the definition of differentiability.

Theorem 14 (Corollary of 4.4) *Suppose $X \subseteq \mathbf{R}^n$ is open and $x \in X$. If $f : X \to \mathbf{R}^m$ is C^2, then*

$$f(x+h) = f(x) + Df(x)h + O\left(|h|^2\right) \text{ as } h \to 0$$

Quadratic Terms:

We treat each component of the function separately, so consider $f : X \to \mathbf{R}$, $X \subseteq \mathbf{R}^n$ an open set. Let

$$\begin{aligned}
D^2 f(x) &= \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix} \\
f \in C^2 &\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x) \\
&\Rightarrow D^2 f(x) \text{ is symmetric} \\
&\Rightarrow D^2 f(x) \text{ has an orthonormal basis of eigenvectors} \\
&\qquad \text{and thus can be diagonalized}
\end{aligned}$$

Theorem 15 (Stronger Version of Thm. 4.4) *Let $X \subseteq \mathbf{R}^n$ be open, $f : X \to \mathbf{R}$, $f \in C^2(X)$, and $x \in X$. Then*

$$f(x+h) = f(x) + Df(x)h + \frac{1}{2}h^\top (D^2 f(x))h + o\left(|h|^2\right) \text{ as } h \to 0$$

If $f \in C^3$,

$$f(x+h) = f(x) + Df(x)h + \frac{1}{2}h^\top (D^2 f(x))h + O\left(|h|^3\right) \text{ as } h \to 0$$

Remark: de la Fuente assumes X is convex. X is said to be *convex* if, for every $x, y \in X$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in X$. Notice we don't need this. Since X is open,

$$x \in X \Rightarrow \exists \delta > 0 \text{ s.t. } B_\delta(x) \subseteq X$$

and $B_\delta(x)$ is convex.

Definition 16 We say f has a *saddle* at x if $Df(x) = 0$ but f has neither a local maximum nor a local minimum at x.

Corollary 17 *Suppose $X \subseteq \mathbf{R}^n$ is open and $x \in X$. If $f : X \to \mathbf{R}$ is C^2, then there is an orthonormal basis $\{v_1, \ldots, v_n\}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ of $D^2 f(x)$ such that*

$$\begin{aligned} f(x+h) &= f(x + \gamma_1 v_1 + \cdots + \gamma_n v_n) \\ &= f(x) + \sum_{i=1}^n (Df(x)v_i)\,\gamma_i + \frac{1}{2}\sum_{i=1}^n \lambda_i \gamma_i^2 + o\left(|\gamma|^2\right) \end{aligned}$$

where $\gamma_i = h \cdot v_i$.

1. *If $f \in C^3$, we may strengthen $o(|\gamma|^2)$ to $O(|\gamma|^3)$.*
2. *If f has a local maximum or local minimum at x, then*

$$Df(x) = 0$$

3. *If $Df(x) = 0$, then*

$$\begin{aligned} \lambda_1, \ldots, \lambda_n > 0 &\Rightarrow f \textit{ has a local minimum at } x \\ \lambda_1, \ldots, \lambda_n < 0 &\Rightarrow f \textit{ has a local maximum at } x \\ \lambda_i < 0 \textit{ for some } i,\ \lambda_j > 0 \textit{ for some } j &\Rightarrow f \textit{ has a saddle at } x \\ \lambda_1, \ldots, \lambda_n \geq 0,\ \lambda_i > 0 \textit{ for some } i &\Rightarrow f \textit{ has a local minimum} \\ & \quad\ \textit{or a saddle at } x \\ \lambda_1, \ldots, \lambda_n \leq 0,\ \lambda_i < 0 \textit{ for some } i &\Rightarrow f \textit{ has a local maximum} \\ & \quad\ \textit{or a saddle at } x \\ \lambda_1 = \cdots = \lambda_n = 0 & \qquad \textit{gives no information.} \end{aligned}$$

Proof: (Sketch) From our study of quadratic forms, we know the behavior of the quadratic terms is determined by the signs of the eigenvalues. If $\lambda_i = 0$ for some i, then we know that the quadratic form arising from the second partial derivatives is identically zero in the direction v_i, and the higher derivatives will determine the behavior of the function f in the direction v_i. For example, if $f(x) = x^3$, then $f'(0) = 0$, $f''(0) = 0$, but we know that f has a saddle at $x = 0$; however, if $f(x) = x^4$, then again $f'(0) = 0$ and $f''(0) = 0$ but f has a local (and global) minimum at $x = 0$.■

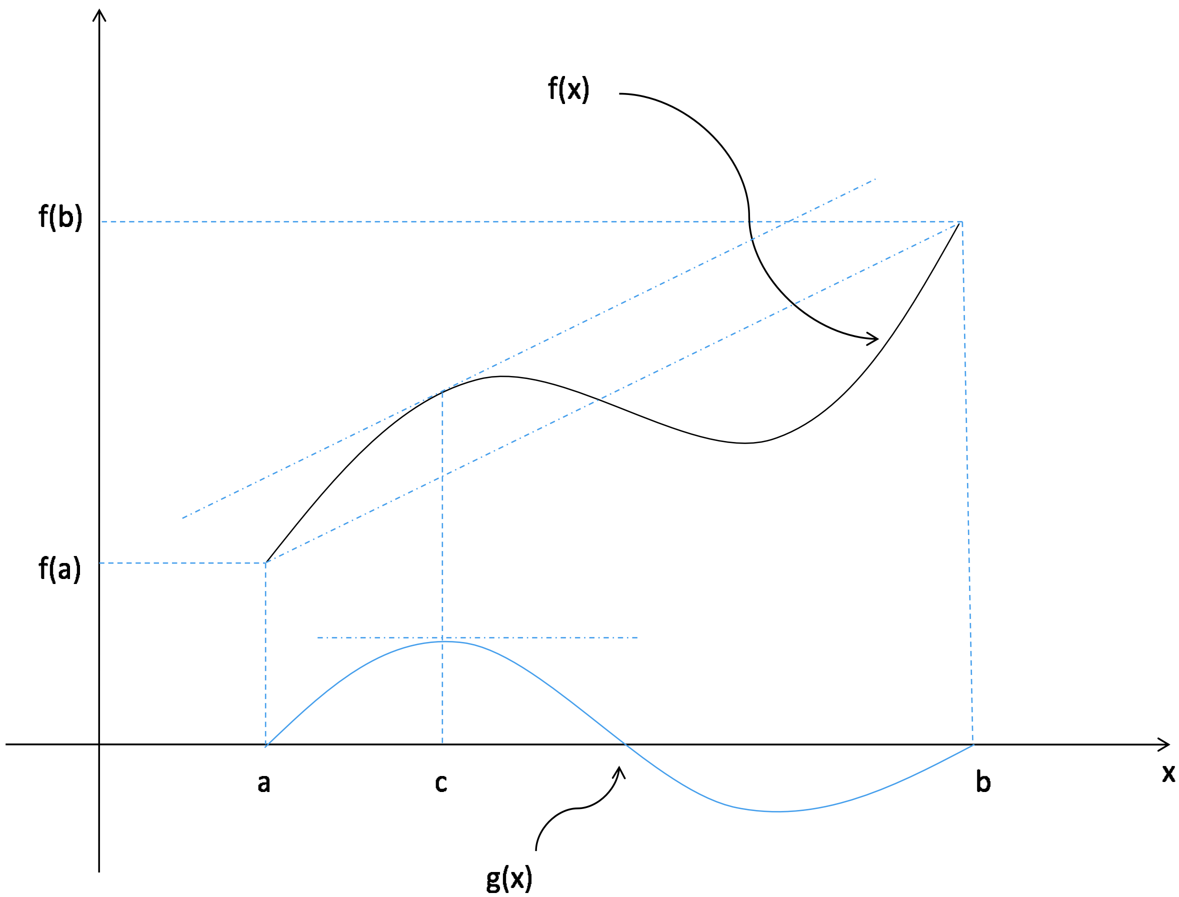

Figure 1: The Mean Value Theorem.