Economics 204 Summer/Fall 2023
Lecture 12–Tuesday August 8, 2023

Inverse and Implicit Function Theorems, and Generic Methods:

In this lecture we develop some of the most important concepts and tools for comparative statics. In many problems we are interested in how endogenously determined variables are affected by exogenously given parameters. Here we study problems in which the variables of interest are characterized as solutions to a parameterized family of equations.

To formalize, let $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ be open, and let $f : X \times A \to \mathbf{R}^m$. For a given $a \in A$, consider solutions $x \in X$ to the family of equations

$$f(x, a) = 0$$

We want to characterize the set of solutions and study how this set depends on the parameter a.

We start with a simple example, to which we return throughout the lecture. Consider the function $f : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ defined by

$$f(x, a) = \sin x + a$$

Let $X = (0, 2\pi)$. For fixed a, let $f_a(x) = f(x, a) = \sin x + a$ denote the perturbed function. We are interested in the solutions $x \in (0, 2\pi)$ to the equation

$$f_a(x) = f(x, a) = \sin x + a = 0$$

that is, the $x \in (0, 2\pi)$ such that

$$\sin x = -a$$

Let $\Psi : A \to 2^X$ denote the solution correspondence, so

$$\Psi(a) = \{x \in (0, 2\pi) : f_a(x) = \sin x + a = 0\}$$

Start with $a = 0$. For $x \in (0, 2\pi)$,

$$f_0(x) = \sin x = 0 \iff x = \pi$$

so $\Psi(0) = \{\pi\}$. Notice that for x near π, for example in the neighborhood $(\pi/2, 3\pi/2)$, and for a near 0, $\sin^{-1}(a)$ remains single-valued and depends smoothly on a. In addition, we can predict the direction of change: x is increasing in a. See Figure 1.

Now consider $a = 1$. For $x \in (0, 2\pi)$,

$$\begin{aligned} f_1(x) &= \sin x + 1 = 0 \\ \iff &\sin x = -1 \\ \iff &x = \frac{3\pi}{2} \end{aligned}$$

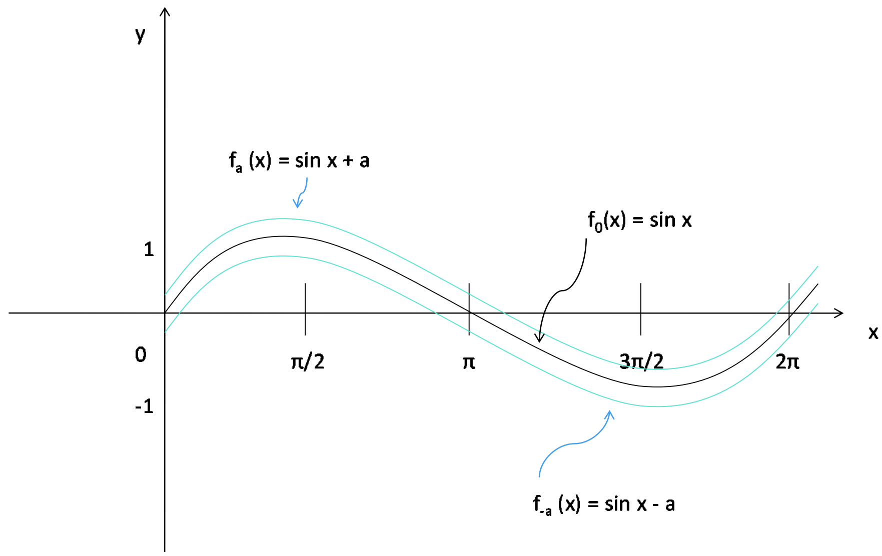

Figure 1: Near $x = \pi$ and $a = 0$ there is a unique solution to $\sin x + a = 0$, which is increasing in a.

So $\Psi(1) = \{3\pi/2\}$. But note that for $a' > 1$, $\Psi(a') = \emptyset$, while for $a < 1$ close to 1, there are two solutions near $3\pi/2$, one above and one below $3\pi/2$. Ψ is not lower hemicontinuous at $a = 1$; see Figure 2. In this case, comparative statics predictions are problematic.

What distinguishes problems for which local comparative statics predictions are possible, as in the case $a = 0$, from those for which comparative statics are impossible, as in the case $a = 1$? The results of this lecture will provide some techniques for answering these questions.

Section 4.3 (Conclusion). Regular and Critical Points and Values

We start with an accounting point.

Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \to \mathbf{R}^m$ is differentiable at $x \in X$, and let $W = \{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbf{R}^n$. Then $df_x \in L(\mathbf{R}^n, \mathbf{R}^m)$, and

$$\begin{aligned} \operatorname{Rank} df_x &= \dim \operatorname{Im}(df_x) \\ &= \dim \operatorname{span} \{df_x(e_1), \ldots, df_x(e_n)\} \\ &= \dim \operatorname{span} \{Df(x)e_1, \ldots, Df(x)e_n\} \\ &= \dim \operatorname{span} \{\text{column } 1 \text{ of } Df(x), \ldots, \text{column } n \text{ of } Df(x)\} \\ &= \operatorname{Rank} Df(x) \end{aligned}$$

Thus,

$$\operatorname{Rank} df_x \leq \min\{m, n\}$$

We say df_x has *full rank* if $\operatorname{Rank} df_x = \min\{m, n\}$, that is, is df_x has maximum possible rank.

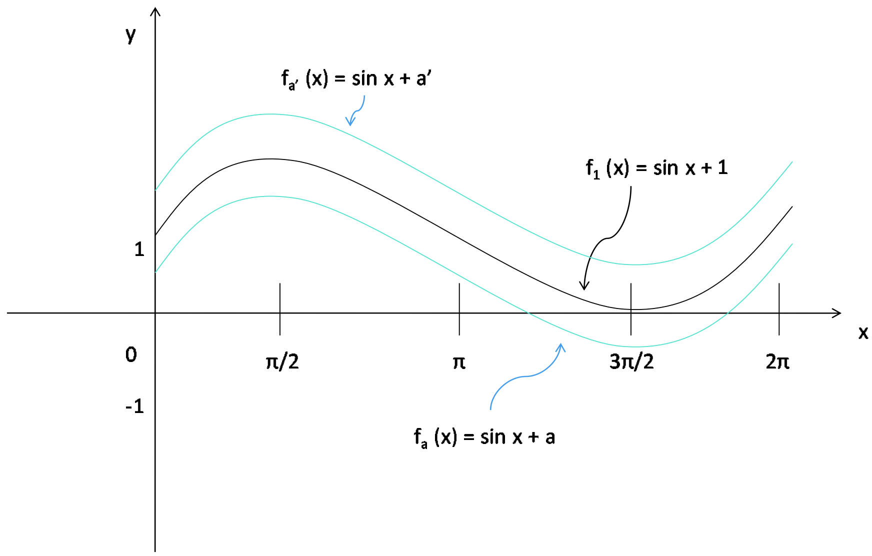

Figure 2: Near $x = \frac{3\pi}{2}$ and $a = 1$, there is not a unique solution to $\sin x + a = 0$ and the solution correspodence is not lower hemicontinuous in a.

Definition 1 Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \to \mathbf{R}^m$ is differentiable on X.

- x is a *regular point* of f if $\text{Rank}\, df_x = \min\{m, n\}$.
- x is a *critical point* of f if $\text{Rank}\, df_x < \min\{m, n\}$.
- y is a *critical value* of f if there exists $x \in f^{-1}(y)$ such that x is a critical point of f.
- y is a *regular value* of f if y is not a critical value of f

Remark: The definition of regular point and critical point above are the standard ones. In de la Fuente (as well as in Mas-Colell, Whinston, and Green) a different definition is given, in which m is used in place of $\min\{m, n\}$. This implicitly assumes that $m \le n$. Notice that if $m \le n$, so the domain of f has dimension greater than the range, then the two are equivalent. If instead $m > n$, so the domain of f has dimension smaller than the range, then since $\text{Rank}\, df_x \le \min\{m, n\} = n < m$, every $x \in X$ will be a critical point in the de la Fuente and MWG definitions, and every $y \in f(X)$ will be a critical value. In contrast, the definition above labels a point regular if $Df(x)$ has maximal rank, and critical otherwise.

Remark: Notice that if $y \notin f(X)$, so $f^{-1}(y) = \emptyset$, then y is automatically a regular value of f.

Example: Consider the function $g : (0, 2\pi) \to \mathbf{R}$ defined by

$$g(x) = \sin x$$

Then $g = f_0$ from our opening example. Note that $g'(x) = \cos x$, so $g'(x) = 0$ for $x = \pi/2$ and $x = 3\pi/2$. $Dg(x)$ is the 1×1 matrix $(g'(x))$, so $\text{Rank}\, dg_x = \text{Rank}\, Dg(x) = 1$ if and only if $g'(x) \neq 0$. Thus, the critical points of g are $\pi/2$ and $3\pi/2$, and the set of regular points of g is

$$\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$$

The critical values of g are $g(\pi/2) = \sin(\pi/2) = 1$ and $g(3\pi/2) = \sin(3\pi/2) = -1$, and the set of regular values of g is

$$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

In particular, notice that 0 is not a critical value of g.

Given $a \in \mathbf{R}$, as above consider the perturbed function

$$f_a(x) = g(x) + a$$

Notice that $f_a'(x) = g'(x)$, so the critical points of f_a are the same as those of g, $\pi/2$ and $3\pi/2$.

For a close to zero, the solution to the equation

$$f_a(x) = 0$$

near $x = \pi$ moves smoothly with respect to changes in a. Notice that the direction the solution moves is determined by the sign of f_a'.

Now let $a = 1$. Since $3\pi/2$ is a critical point of f_1, 0 is a critical value of f_1.

Inverse Function Theorem:

Theorem 2 (Thm. 4.6, Inverse Function Theorem) *Suppose $X \subseteq \mathbf{R}^n$ is open, $f : X \to \mathbf{R}^n$ is C^1 on X, and $x_0 \in X$. If $\det Df(x_0) \neq 0$ (i.e. x_0 is a regular point of f) then there are open neighborhoods U of x_0 and V of $f(x_0)$ such that*

$$\begin{array}{rcl} f : U \to V & \textit{is} & \textit{one-to-one and onto} \\ f^{-1} : V \to U & \textit{is} & C^1 \\ Df^{-1}(f(x_0)) & = & [Df(x_0)]^{-1} \end{array}$$

If in addition $f \in C^k$, then $f^{-1} \in C^k$.

Proof: Read the proof in de la Fuente. This is pretty hard. The idea is that since $\det Df(x_0) \neq 0$, then $df_{x_0} : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one and onto. You need to find a neighborhood U of x_0 sufficiently small such that the Contraction Mapping Theorem implies that f is one-to-one and onto.

To see the formula for Df^{-1}, let id_U denote the identity function from U to U and I denote the $n \times n$ identity matrix. Then

$$\begin{aligned} Df^{-1}(f(x_0))Df(x_0) &= D(f^{-1} \circ f)(x_0) \\ &= D(\,\text{id}_U(x_0)) \\ &= I \\ \Rightarrow Df^{-1}(f(x_0)) &= [Df(x_0)]^{-1} \end{aligned}$$

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Remark: f is one-to-one only on U; it need not be one-to-one globally. Thus f^{-1} is only a local inverse.

Example: Let $g : (0, 2\pi) \to \mathbf{R}$ be given by $g(x) = \sin x$ as above, and let $x_0 = \pi$. Then $g'(x_0) = \cos\pi = -1 \neq 0$, so by the inverse function theorem there exists an open set $U \subseteq (0, 2\pi)$ with $\pi \in U$, an open set $V \subseteq \mathbf{R}$ with $0 = g(\pi) \in V$ and a C^1 function $h : V \to U$ such that $g(h(v)) = v$ for all $v \in V$.

Notice that at $x = 3\pi/2$, $g'(x) = \cos(3\pi/2) = 0$, and g has no local inverse function there: for every open neighborhood U of $3\pi/2$ and every open neighborhood V of $-1 = g(3\pi/2)$, there exists $v \in V$ and $x_1 \neq x_2 \in U$ such that $g(x_1) = \sin x_1 = v = \sin x_2 = g(x_2)$.

Remark: Read Section 4.5 on your own.

Section 5.2. Implicit Function Theorem

Theorem 3 (Thm. 2.2, Implicit Function Theorem) *Suppose $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ are open and $f : X \times A \to \mathbf{R}^n$ is C^1. Suppose $f(x_0, a_0) = 0$ and $\det(D_x f(x_0, a_0)) \neq 0$, i.e. x_0 is a regular point of $f(\cdot, a_0)$. Then there are open neighborhoods U of x_0 ($U \subseteq X$) and W of a_0 such that*

$$\forall a \in W \quad \exists! x \in U \text{ s.t. } f(x, a) = 0$$

For each $a \in W$ let $g(a)$ be that unique x. Then $g : W \to X$ is C^1 and

$$Dg(a_0) = -\left[D_x f(x_0, a_0)\right]^{-1}\left[D_a f(x_0, a_0)\right]$$

If in addition $f \in C^k$, then $g \in C^k$.

Proof: Use the Inverse Function Theorem in the right way. Why is the formula for Dg correct? Assuming the implicit function exists and is differentiable,

$$\begin{aligned} 0 &= Df(g(a), a)(a_0) \\ &= D_x f(x_0, a_0) Dg(a_0) + D_a f(x_0, a_0) \\ Dg(a_0) &= -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0) \end{aligned}$$

The following argument outlines the proof that g is differentiable:

$$\begin{aligned} f(x_0, a_0 + h) &= f(x_0, a_0) + D_a f(x_0, a_0)h + o(h) \\ &= D_a f(x_0, a_0)h + o(h) \end{aligned}$$

Solve for Δx that brings f back to zero:

$$\begin{aligned} 0 &= f(x_0 + \Delta x, a_0 + h) \\ &= f(x_0, a_0 + h) + D_x f(x_0, a_0 + h)\Delta x + o(\Delta x) \\ &= f(x_0, a_0) + D_a f(x_0, a_0)h + D_x f(x_0, a_0 + h)\Delta x + o(\Delta x) + o(h) \\ &= D_a f(x_0, a_0)h + D_x f(x_0, a_0 + h)\Delta x + o(\Delta x) + o(h) \\ D_x f(x_0, a_0 + h)\Delta x &= -D_a f(x_0, a_0)h + o(\Delta x) + o(h) \end{aligned}$$

Because f is C^1 and the determinant is a continuous functions of the entries of the matrix, $\det D_x f(x_0, a_0 + h) \neq 0$ for h sufficiently small, so

$$\begin{aligned} \Delta x &= -\left[D_x f(x_0, a_0 + h)\right]^{-1} D_a f(x_0, a_0)h + o(\Delta x) + o(h) \\ &= -\left[D_x f(x_0, a_0) + o(1)\right]^{-1} D_a f(x_0, a_0)h + o(\Delta x) + o(h) \text{ since } f \in C^1 \\ &= -\left[D_x f(x_0, a_0)\right]^{-1} D_a f(x_0, a_0)h + o(\Delta x) + o(h) \text{ since } f \in C^1 \end{aligned}$$

Then

$$\begin{aligned} |\Delta x + o(\Delta x)| &= O(h) \\ \Rightarrow |\Delta x| &= O(h) \\ \Rightarrow o(\Delta x) &= o(h) \\ \Rightarrow \Delta x &= -\left[D_x f(x_0, a_0)\right]^{-1} D_a f(x_0, a_0)h + o(h) \end{aligned}$$

By the definition of the derivative,

$$Dg(a_0) = -\left[D_x f(x_0, a_0)\right]^{-1} D_a f(x_0, a_0)$$

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Corollary 4 *Suppose $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ are open and $f : X \times A \to \mathbf{R}^n$ is C^1. If 0 is a regular value of $f(\cdot, a_0)$, then the correspondence*

$$a \mapsto \{x \in X : f(x, a) = 0\}$$

is lower hemicontinuous at a_0.

Proof: If 0 is a regular value of $f(\cdot, a_0)$, then given any $x_0 \in \{x \in X : f(x, a_0) = 0\}$, we can find a local implicit function g; in other words, if a is sufficiently close to a_0, then $g(a) \in \{x \in X : f(x, a) = 0\}$; the continuity of g then shows that the correspondence $\{x \in X : f(x, a) = 0\}$ is lower hemicontinuous at a_0. ■

Example: Again we return to our opening example: $f : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ defined by $f(x, a) = \sin x + a$. Let $x_0 = \pi$ and $a_0 = 0$. Then $f(x_0, a_0) = \sin \pi = 0$ and $D_x f(x_0, a_0) = \cos \pi = -1 \neq 0$. So $x_0 = \pi$ is a regular point of $f(\cdot, a_0)$. By the Implicit Function Theorem, there are open neighborhoods U containing π and W containing 0 and a C^1 function $h : W \to U$ such that $f(h(a), a) = 0$ for every $a \in W$ and such that

$$Dh(a_0) = -[\cos \pi]^{-1} \cdot 1 = 1$$

So the local solution is increasing in a near a_0 (as we saw above).

Again notice that at $x = 3\pi/2$ and $a = 1$, $D_x f(x, a) = 0$ and no local implicit function exists: for every open neighborhood U of $3\pi/2$ and W of 1, for any $a' > 1$ there are no $x' \in U$ such that $f(x', a') = \sin x' + a' = 0$.

Transversality and Genericity

Definition 5 Suppose $A \subseteq \mathbf{R}^n$. A has *Lebesgue measure zero* if for every $\varepsilon > 0$ there is a countable collection of rectangles $I_1, I_2, \ldots$ such that

$$\sum_{k=1}^{\infty} \mathrm{Vol}\,(I_k) < \varepsilon \text{ and } A \subseteq \bigcup_{k=1}^{\infty} I_k$$

Here by a rectangle we mean $I_k = \times_{j=1}^{n} (a_j^k, b_j^k)$ for some $a_j^k < b_j^k \in \mathbf{R}$, and

$$\mathrm{Vol}\ (I_k) = \prod_{j=1}^{n} |b_j^k - a_j^k|$$

Notice that this defines Lebesgue measure zero without defining Lebesgue measure.

Examples:

1. "Lower-dimensional" sets have Lebesgue measure zero. For example,

   $$A = \{x \in \mathbf{R}^2 : x_2 = 0\}$$

   has measure zero. See Figure 3.

2. Any finite set has Lebesgue measure zero in $\mathbf{R}^n$.

3. If A_n has Lebesgue measure zero $\forall n$ then $\cup_{n \in \mathbf{N}} A_n$ has Lebesgue measure zero.

4. $\mathbf{Q}$ and every countable set has Lebesgue measure zero.

5. No open set in $\mathbf{R}^n$ has Lebesgue measure zero.

   If $O \subset \mathbf{R}^n$ is open, then there exists a rectangle R such that $\bar{R} \subseteq O$ and such that $\mathrm{Vol}\ (R) = r > 0$. If $\{I_j\}$ is any collection of rectangles such that $O \subseteq \cup_{j=1}^{\infty} I_j$, then $\bar{R} \subseteq O \subseteq \cup_{j=1}^{\infty} I_j$, so $\sum_{j=1}^{\infty}\ \mathrm{Vol}\ (I_j) \geq\ \mathrm{Vol}\ (R) = r > 0$. See Figure 4.

This is a natural formulation of the notion that A is a small set. Without specifying a probability measure explicitly, this expresses the idea that if $x \in \mathbf{R}^n$ is chosen at random, the probability that $x \in A$ is zero.

A function may have many critical points; for example, if a function is constant on an interval, then every element of the interval is a critical point. But it can't have many critical *values.*[1]

Theorem 6 (Thm. 2.4, Sard's Theorem) *Let $X \subseteq \mathbf{R}^n$ be open, and $f : X \to \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n-m\}$. Then the set of all critical values of f has Lebesgue measure zero.*

Proof: First, we give a false proof that conveys the essential idea as to why the theorem is true; it can be turned into a correct proof. Suppose $m = n$. Let C be the set of critical points of f, V the set of critical values. Then

$$\begin{aligned} \mathrm{Vol}\,(V) &= \mathrm{Vol}\,(f(C)) \\ &\leq \int_C |\det Df(x)|\,dx \text{ (equality if } f \text{ is one-to-one)} \\ &= \int_C 0\,dx \\ &= 0 \end{aligned}$$

Now, we outline how to turn this into a proof. First, show that we can write $X = \cup_{j \in \mathbf{N}} X_j$, where each X_j is a compact subset of $[-j, j]^n$. Let $C_j = C \cap X_j$. Fix j for now. Since f is C^1,

$$\begin{aligned} x_k \to x &\Rightarrow \det Df(x_k) \to \det Df(x) \\ \{x_k\} \subseteq C_j, x_k \to x &\Rightarrow \det Df(x) = 0 \Rightarrow x \in C_j \end{aligned}$$

so C_j is closed, hence compact. Since X is open and C_j is compact, there exists $\delta_1 > 0$ such that

$$B_{\delta_1}[C_j] = \cup_{x \in C_j} B_{\delta_1}[x] \subseteq X$$

$B_{\delta_1}[C_j]$ is bounded, and, using the compactness of C_j, one can show it is closed, so it is compact. Since $\det Df(x)$ is continuous on $B_{\delta_1}[C_j]$, it is uniformly continuous on $B_{\delta_1}[C_j]$. Then given $\varepsilon > 0$, we can find $\delta \leq \delta_1$ such that $B_\delta[C_j] \subseteq [-2j, 2j]^n$ and

$$x \in B_\delta[C_j] \Rightarrow |\det Df(x)| \leq \frac{\varepsilon}{2 \cdot 4^n j^n}$$

Then

$$\begin{aligned} f(C_j) &\subseteq f(B_\delta[C_j]) \\ \mathrm{Vol}\,(f(B_\delta[C_j])) &\leq \int_{[-2j,2j]^n} \frac{\varepsilon}{2 \cdot 4^n j^n}\,dx \\ &= \frac{\varepsilon}{2} \end{aligned}$$

[1]If $m > n$, then every $x \in X$ is critical using de la Fuente's definition, because $\mathrm{Rank}\,Df(x) \leq n < m$. Thus, every $y \in f(X)$ is a critical value, using de la Fuente's definition. This does not contradict Sard's Theorem, since one can show that $f(X)$ is a set of Lebesgue measure zero when $m > n$ and $f \in C^1$.

Since f is C^1, show that $f(C_j)$ can be covered by a countable collection of rectangles of total volume less than ε. Since $\varepsilon > 0$ is arbitrary, $f(C_j)$ has Lebesgue measure zero. Then

$$f(C) = f\left(\cup_{j\in\mathbf{N}} C_j\right) = \cup_{n\in\mathbf{N}} f(C_j)$$

is a countable union of sets of Lebesgue measure zero, so $f(C)$ has Lebesgue measure zero. ■

Remark: Sard's Theorem has a number of powerful implications. Given a randomly chosen function f, it is very unlikely that zero will be a critical value of f. If by some fluke zero *is* a critical value of f, then a slight perturbation of f will make zero a regular value. We return to a more wide-ranging version of this statement below.

Example: Let $g : (0, 2\pi) \to \mathbf{R}$ be given by $g(x) = \sin x$. We calculated by hand above that the set of critical values of g is $\{-1, 1\}$. Since this set is finite, it has Lebesgue measure zero.

Let $g : \mathbf{R}^n \to \mathbf{R}^n$ be C^1. Consider the family of n equations in n variables:

$$g(x) = 0$$

For example, consider Figure 5. Here for some x such that $g(x) = 0$, $\text{rank}\,(Dg(x)) < n$. That is, some $x \in g^{-1}(0)$ is a critical point of g, thus 0 is a critical value of g. By Sard's Theorem, almost every $a \neq 0$ is a regular value of g. So for almost every a, that is for a outside a set of Lebesgue measure 0, $Dg(x)$ has full rank for every $x \in g^{-1}(a)$, that is, for every x solving $g(x) = a$. For any such a and any $x \in g^{-1}(a)$, we can use the Inverse Function Theorem to show that a local inverse $x(a)$ exists, and give a formula for $Dx(a)$.

We can rephrase this observation by thinking about a family of equations indexed by a set of parameters as follows. Let $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be given by

$$f(x, a) = g(x) - a$$

By Sard's Theorem, there exists a set A of Lebesgue measure zero such that for each $a \notin A$, $D_x f(x, a)$ has full rank k at every x such that $f(x, a) = 0$. Thus 0 is a regular value of $f(\cdot, a)$ for every $a \notin A$.

Fix $a^* \notin A$ and x^* such that $f(x^*, a^*) = 0$. By the Implicit Function Theorem, there exist open sets A^* containing a^* and X^* containing x^*, and a C^1 function $x : A^* \to X^*$ such that

- $x(a^*) = x^*$
- $f(x(a), a) = 0$ for every $a \in A^*$
- if $(x, a) \in X^* \times A^*$ then
  $$f(x, a) = 0 \iff x = x(a)$$
  that is, x^* is locally unique, and $x(a)$ is locally unique for each $a \in A^*$

- $Dx(a^*) = -[D_x f(x^*, a^*)]^{-1} D_a f(x^*, a^*)$

We would like a more general result along these lines that allows for a richer class of parameterizations in which parameters need not enter linearly, the parameter set may have dimension different from the set of variables x, and the number of equations m might differ from the number of endogenous variables n. This is what the Transversality Theorem gives.

Suppose $f : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^m$ is C^1. We are interested in studying the parameterized family of equations

$$f(x, a) = 0$$

where, as above, we interpret $a \in \mathbf{R}^p$ to be a vector of parameters that indexes the function $f(\cdot, a)$. For a given a, we are interested in the set of solutions $\{x \in X : f(x, a) = 0\}$ and the way that this correspondence depends on a. If f is separable in a, that is, $f(x, a) = g(x) + a$, then we can use Sard's Theorem, as above, but separability is not required. If f depends on a in a nonseparable fashion, it is enough that from any solution $f(x, a) = 0$, any directional change in f can be achieved by arbitrarily small changes in x and a. This is formalized in the Transversality Theorem.

Theorem 7 (Thm. 2.5', Transversality Theorem) *Let $X \subseteq \mathbf{R}^n$ and $A \subseteq \mathbf{R}^p$ be open, and $f : X \times A \to \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n - m\}$. Suppose that 0 is a regular value of f. Then there is a set $A_0 \subseteq A$ such that $A \setminus A_0$ has Lebesgue measure zero and for all $a \in A_0$, 0 is a regular value of $f_a = f(\cdot, a)$.*

Remark: Notice the important difference between the statement that 0 is a regular value of f (one of the assumptions of the Transversality Theorem), and the statement that 0 is a regular value of f_a for a fixed $a \in A_0$ (part of the conclusion of the Transversality Theorem). 0 is a regular value of f if and only if $Df(x, a)$ has full rank for every (x, a) such that $f(x, a) = 0$. Instead, for fixed $a_0 \in A_0$, 0 is a regular value of $f_{a_0} = f(\cdot, a_0)$ if and only if $D_x f(x, a_0)$ has full rank for every x such that $f_{a_0}(x) = f(x, a_0) = 0$.

Remark: Consider the important special case in which $n = m$, so we have as many equations (m) as endogenous variables (n). Notice that this is also the case in the example above with linear perturbations that we analyzed using Sard's Theorem. In this case, with $n = m$, suppose f is C^1 (note that $1 = 1 + \max\{0, n - n\}$). If 0 is a regular value of f, that is, $Df(x, a)$ has rank $n = m$ for every (x, a) such that $f(x, a) = 0$, then by the Transversality Theorem there is a set $A_0 \subset A$ such that $A \setminus A_0$ has Lebesgue measure zero and for every $a_0 \in A_0$, $D_x f(x, a_0)$ has rank $n = m$ for all x such that $f(x, a_0) = 0$. Fix $a_0 \in A_0$ and x_0 such that $f(x_0, a_0) = 0$. As above, by the Implicit Function Theorem, there exist open sets A^* containing a_0 and X^* containing x_0, and a C^1 function $x : A^* \to X^*$ such that

- $x(a_0) = x_0$
- $f(x(a), a) = 0$ for every $a \in A^*$

- if $(x,a) \in X^* \times A^*$ then
$$f(x,a) = 0 \iff x = x(a)$$
that is, x_0 is locally unique, and $x(a)$ is locally unique for each $a \in A^*$

- $Dx(a_0) = -[D_x f(x_0,a_0)]^{-1} D_a f(x_0,a_0)$

Example: Again we return to the opening example: $f : (0,2\pi) \times \mathbf{R} \to \mathbf{R}$ defined by $f(x,a) = \sin x + a$. Then for any (x,a) such that $f(x,a) = 0$, $Df(x,a) = (\cos x, 1)$ which has rank $1 = \min\{2,1\}$. Thus 0 is a regular value of f. (Notice we have shown something stronger, that any value c is a regular value of f, since we did not use the fact that $f(x,a) = \sin x + a = 0$.) Set $A_0 = \mathbf{R} \setminus \{-1,1\}$. Since $\{-1,1\}$ is a finite set, it has Lebesgue measure zero in $\mathbf{R}$. Again we have already calculated by hand that for any $a \in A_0$, 0 is a regular value of $f_a = f(\cdot, a)$.

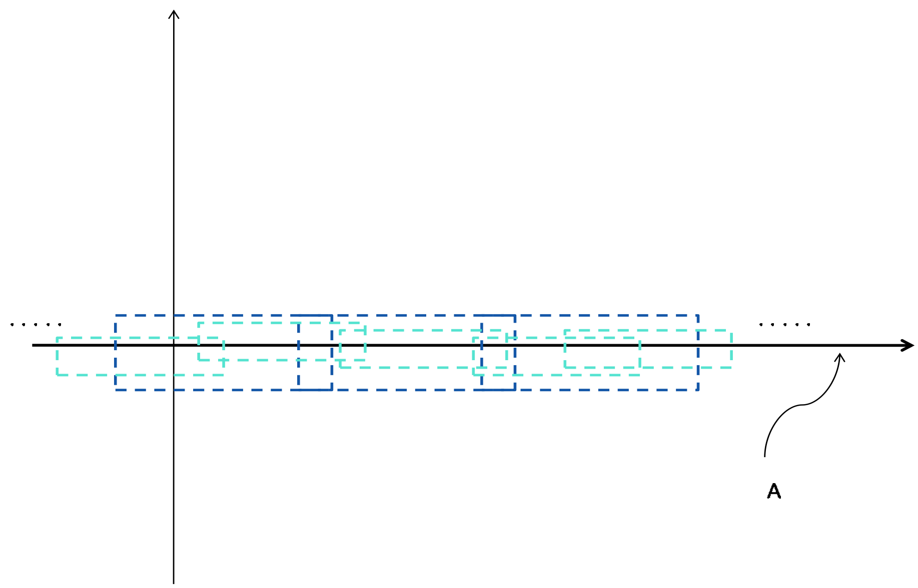

Figure 3: The set $A = \{x \in \mathbf{R}^2 : x_2 = 0\}$ has Lebesgue measure zero in $\mathbf{R}^2$.

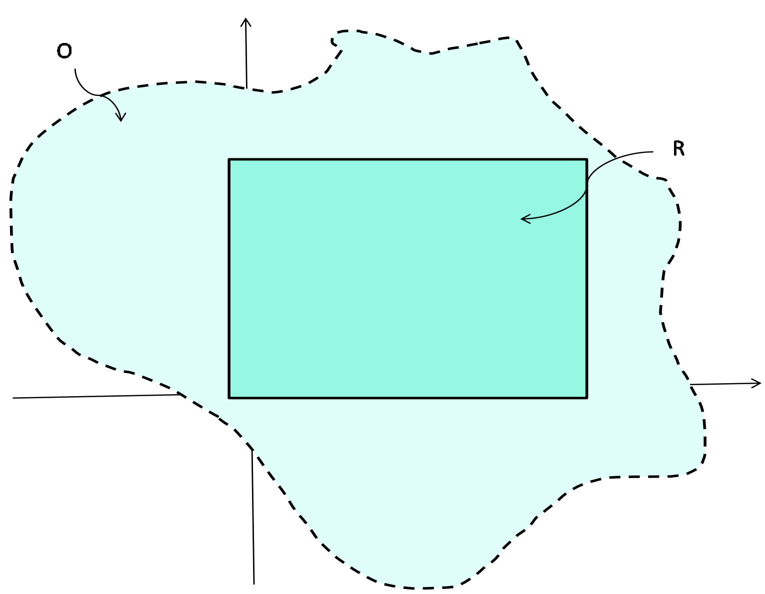

Figure 4: An open set O does not have Lebesgue measure zero.

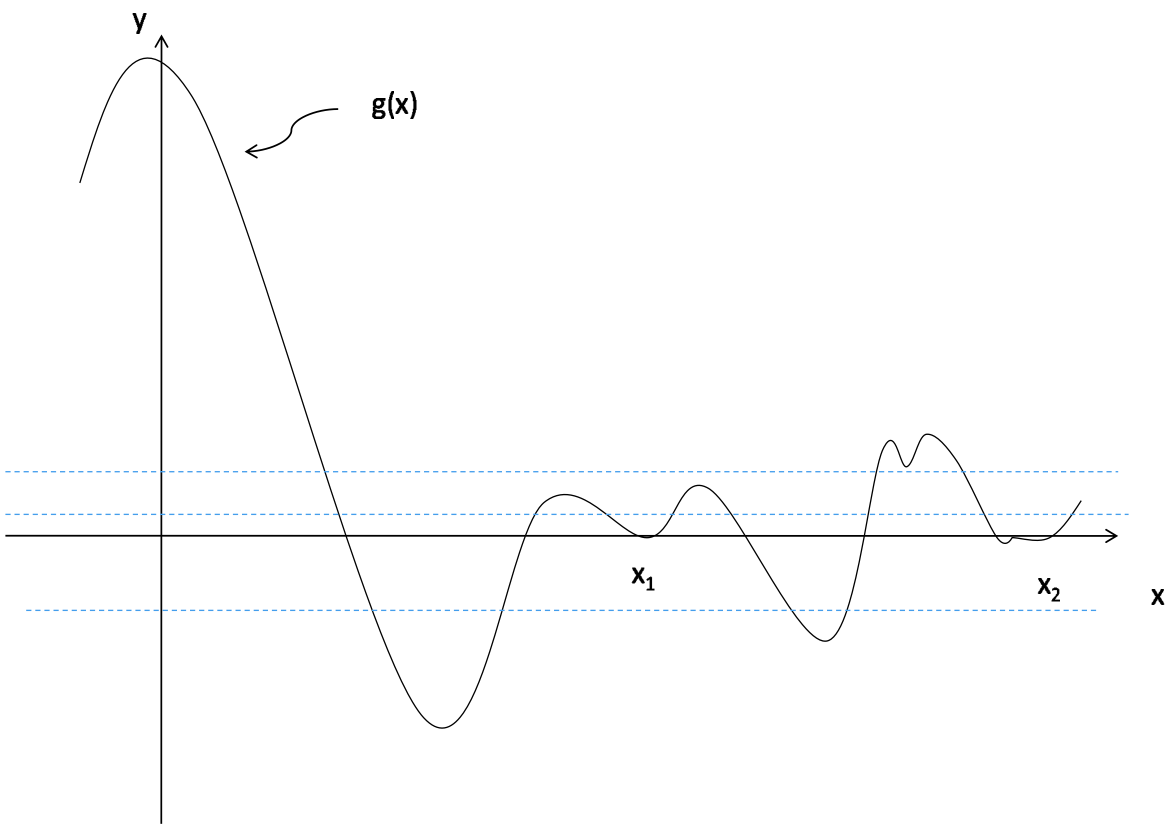

Figure 5: x_1 and x_2 are critical points of g. Almost every value $a \neq 0$ is a regular value of g.