Economics 204 Summer/Fall 2023
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Chapter 3. Linear Algebra

Section 3.1. Bases

Definition 1 Let X be a vector space over a field F. A *linear combination* of $x_1, \ldots, x_n \in X$ is a vector of the form

$$y = \sum_{i=1}^{n} \alpha_i x_i \text{ where } \alpha_1, \ldots, \alpha_n \in F$$

α_i is the *coefficient* of x_i in the linear combination.

If $V \subseteq X$, the *span* of V, denoted span V, is the set of all linear combinations of elements of V. The set $V \subseteq X$ *spans* X if span $V = X$.

Definition 2 A set $V \subseteq X$ is *linearly dependent* if there exist $v_1, \ldots, v_n \in V$ and $\alpha_1, \ldots, \alpha_n \in F$ not all zero such that

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

A set $V \subseteq X$ is *linearly independent* if it is not linearly dependent.

Thus $V \subseteq X$ is linearly independent if and only if

$$\sum_{i=1}^{n} \alpha_i v_i = 0, \quad v_i \in V \ \forall i \Rightarrow \alpha_i = 0 \ \forall i$$

Definition 3 A *Hamel basis* (often just called a *basis*) of a vector space X is a linearly independent set of vectors in X that spans X.

Example: $\{(1,0),(0,1)\}$ is a basis for $\mathbf{R}^2$ (this is the standard basis).

$\{(1,1),(-1,1)\}$ is another basis for $\mathbf{R}^2$: Suppose

$$\begin{aligned}
(x,y) &= \alpha(1,1) + \beta(-1,1) \text{ for some } \alpha, \beta \in \mathbf{R} \\
x &= \alpha - \beta \\
y &= \alpha + \beta \\
x + y &= 2\alpha \\
\Rightarrow \alpha &= \frac{x+y}{2}
\end{aligned}$$

$$\begin{aligned} y - x &= 2\beta \\ \Rightarrow \beta &= \frac{y-x}{2} \\ \Rightarrow (x,y) &= \frac{x+y}{2}(1,1) + \frac{y-x}{2}(-1,1) \end{aligned}$$

Since (x, y) is an arbitrary element of $\mathbf{R}^2$, $\{(1,1),(-1,1)\}$ spans $\mathbf{R}^2$. If $(x, y) = (0, 0)$,

$$\alpha = \frac{0+0}{2} = 0, \quad \beta = \frac{0-0}{2} = 0$$

so the coefficients are all zero, so $\{(1,1),(-1,1)\}$ is linearly independent. Since it is linearly independent and spans $\mathbf{R}^2$, it is a basis.

Example: $\{(1,0,0),(0,1,0)\}$ is not a basis of $\mathbf{R}^3$, because it does not span $\mathbf{R}^3$.

Example: $\{(1,0),(0,1),(1,1)\}$ is not a basis for $\mathbf{R}^2$.

$$1(1,0) + 1(0,1) + (-1)(1,1) = (0,0)$$

so the set is not linearly independent.

Theorem 4 (Thm. 1.2') [1] *Let V be a Hamel basis for X. Then every vector $x \in X$ has a* unique *representation as a linear combination of a finite number of elements of V (with* all *coefficients nonzero).*[2]

Proof: Let $x \in X$. Since V spans X, we can write

$$x = \sum_{s \in S_1} \alpha_s v_s$$

where S_1 is finite, $\alpha_s \in F$, $\alpha_s \neq 0$, and $v_s \in V$ for each $s \in S_1$. Now, suppose

$$x = \sum_{s \in S_1} \alpha_s v_s = \sum_{s \in S_2} \beta_s v_s$$

where S_2 is finite, $\beta_s \in F$, $\beta_s \neq 0$, and $v_s \in V$ for each $s \in S_2$.
Let $S = S_1 \cup S_2$, and define

$$\begin{aligned} \alpha_s = 0 \quad &\text{for} \quad s \in S_2 \setminus S_1 \\ \beta_s = 0 \quad &\text{for} \quad s \in S_1 \setminus S_2 \end{aligned}$$

Then

$$\begin{aligned} 0 &= x - x \\ &= \sum_{s \in S_1} \alpha_s v_s - \sum_{s \in S_2} \beta_s v_s \\ &= \sum_{s \in S} \alpha_s v_s - \sum_{s \in S} \beta_s v_s \\ &= \sum_{s \in S} (\alpha_s - \beta_s) v_s \end{aligned}$$

[1] See Corrections handout.
[2] The unique representation of 0 is $0 = \sum_{i \in \emptyset} \alpha_i b_i$.

Since V is linearly independent, we must have $\alpha_s - \beta_s = 0$, so $\alpha_s = \beta_s$, for all $s \in S$.

$$s \in S_1 \Leftrightarrow \alpha_s \neq 0 \Leftrightarrow \beta_s \neq 0 \Leftrightarrow s \in S_2$$

so $S_1 = S_2$ and $\alpha_s = \beta_s$ for $s \in S_1 = S_2$, so the representation is unique. ■

Theorem 5 *Every vector space has a Hamel basis.*

Proof: The proof uses the Axiom of Choice. Indeed, the theorem is equivalent to the Axiom of Choice. ■

A closely related result, from which you can derive the previous result, shows that any linearly independent set V in a vector space X can be extended to a basis of X.

Theorem 6 *If X is a vector space and $V \subseteq X$ is linearly independent, then there exists a linearly independent set $W \subseteq X$ such that*

$$V \subseteq W \subseteq \operatorname{span} W = X$$

Theorem 7 *Any two Hamel bases of a vector space X have the same cardinality (are numerically equivalent).*

Proof: The proof depends on the so-called Exchange Lemma, whose idea we sketch. Suppose that $V = \{v_\lambda : \lambda \in \Lambda\}$ and $W = \{w_\gamma : \gamma \in \Gamma\}$ are Hamel bases of X. Remove one vector v_{λ_0} from V, so that it no longer spans (if it did still span, then v_{λ_0} would be a linear combination of other elements of V, and V would not be linearly independent). If $w_\gamma \in \operatorname{span}(V \setminus \{v_{\lambda_0}\})$ for every $\gamma \in \Gamma$, then since W spans, $V \setminus \{v_{\lambda_0}\}$ would also span, contradiction. Thus, we can choose $\gamma_0 \in \Gamma$ such that

$$w_{\gamma_0} \notin \text{span } (V \setminus \{v_{\lambda_0}\})$$

Because $w_{\gamma_0} \in \operatorname{span} V$, we can write

$$w_{\gamma_0} = \sum_{i=0}^{n} \alpha_i v_{\lambda_i}$$

where α_0, the coefficient of v_{λ_0}, is not zero (if it were, then we would have $w_{\gamma_0} \in \text{span } (V \setminus \{v_{\lambda_0}\})$). Since $\alpha_0 \neq 0$, we can solve for v_{λ_0} as a linear combination of w_{γ_0} and $v_{\lambda_1}, \ldots, v_{\lambda_n}$, so

$$\begin{aligned} &\text{span } ((V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\}) \\ \supseteq\ &\operatorname{span} V \\ =\ &X \end{aligned}$$

so

$$((V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\})$$

spans X. From the fact that $w_{\gamma_0} \notin \text{span}\ (V \setminus \{v_{\lambda_0}\})$ one can show that

$$((V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\})$$

is linearly independent, so it is a basis of X. Repeat this process to exchange every element of V with an element of W (when V is infinite, this is done by a process called transfinite induction). At the end, we obtain a bijection from V to W, so that V and W are numerically equivalent. ■

Definition 8 The *dimension* of a vector space X, denoted $\dim X$, is the cardinality of any basis of X.

Definition 9 Let X be a vector space. If $\dim X = n$ for some $n \in \mathbf{N}$, then X is *finite-dimensional.* Otherwise, X is *infinite-dimensional.*

Recall that for $V \subseteq X$, $|V|$ denotes the cardinality of the set V.[3]

Example: The set of all $m \times n$ real-valued matrices is a vector space over $\mathbf{R}$. A basis is given by

$$\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

where

$$(E_{ij})_{k\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j \\ 0 & \text{otherwise.} \end{cases}$$

The dimension of the vector space of $m \times n$ matrices is mn.

Theorem 10 (Thm. 1.4) *Suppose* $\dim X = n \in \mathbf{N}$. *If* $V \subseteq X$ *and* $|V| > n$, *then* V *is linearly dependent.*

Proof: If not, so V is linearly independent, then there is a basis W for X that contains V. But $|W| \geq |V| > n = \dim X$, a contradiction. ■

Theorem 11 (Thm. 1.5') *Suppose* $\dim X = n \in \mathbf{N}$ *and* $V \subseteq X$, $|V| = n$.

- *If V is linearly independent, then V spans X, so V is a Hamel basis.*
- *If V spans X, then V is linearly independent, so V is a Hamel basis.*

Proof: (Sketch)

[3] See the Appendix to Lecture 2 for some facts about cardinality.

- If V does not span X, then there is a basis W for X that contains V as a proper subset. Then $|W| > |V| = n = \dim X$, a contradiction.
- If V is not linearly independent, then there is a proper subset V' of V that is linearly independent and for which $\operatorname{span} V' = \operatorname{span} V = X$. But then $|V'| < |V| = n = \dim X$, a contradiction.

■

Note: Read the material on Affine Spaces on your own.

Section 3.2. Linear Transformations

Definition 12 Let X and Y be two vector spaces over the field F. We say $T : X \to Y$ is a *linear transformation* if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall x_1, x_2 \in X, \alpha_1, \alpha_2 \in F$$

Let $L(X, Y)$ denote the set of all linear transformations from X to Y.

Theorem 13 *$L(X, Y)$ is a vector space over F.*

The hard part of proving this theorem is figuring out what you are being asked to prove. Once you figure that out, this is completely trivial, although writing out a complete proof that checks all the vector space axioms is rather tedious. The key is to define scalar multiplication and vector addition, and show that a linear combination of linear transformations is a linear transformation.

Proof: First, define linear combinations in $L(X, Y)$ as follows. For $T_1, T_2 \in L(X, Y)$ and $\alpha, \beta \in F$, define $\alpha T_1 + \beta T_2$ by

$$(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)$$

We need to show that $\alpha T_1 + \beta T_2 \in L(X, Y)$.

$$\begin{aligned}
&(\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2) \\
&= \alpha T_1(\gamma x_1 + \delta x_2) + \beta T_2(\gamma x_1 + \delta x_2) \\
&= \alpha \left(\gamma T_1(x_1) + \delta T_1(x_2)\right) + \beta \left(\gamma T_2(x_1) + \delta T_2(x_2)\right) \\
&= \gamma \left(\alpha T_1(x_1) + \beta T_2(x_1)\right) + \delta \left(\alpha T_1(x_2) + \beta T_2(x_2)\right) \\
&= \gamma \left(\alpha T_1 + \beta T_2\right)(x_1) + \delta \left(\alpha T_1 + \beta T_2\right)(x_2)
\end{aligned}$$

so $\alpha T_1 + \beta T_2 \in L(X, Y)$.

The rest of the proof involves straightforward checking of the vector space axioms. ■

Composition of Linear Transformations

Given $R \in L(X, Y)$ and $S \in L(Y, Z)$, $S \circ R : X \to Z$. We will show that $S \circ R \in L(X, Z)$, that is, the composition of two linear transformations is also linear.

$$\begin{aligned}(S \circ R)(\alpha x_1 + \beta x_2) &= S(R(\alpha x_1 + \beta x_2)) \\ &= S(\alpha R(x_1) + \beta R(x_2)) \\ &= \alpha S(R(x_1)) + \beta S(R(x_2)) \\ &= \alpha (S \circ R)(x_1) + \beta (S \circ R)(x_2)\end{aligned}$$

so $S \circ R \in L(X, Z)$.

Definition 14 Let $T \in L(X, Y)$.

- The *image* of T is $\operatorname{Im} T = T(X)$
- The *kernel* of T is $\ker T = \{x \in X : T(x) = 0\}$
- The *rank* of T is $\operatorname{Rank} T = \dim(\operatorname{Im} T)$

Theorem 15 (Thms. 2.9, 2.7, 2.6: The Rank-Nullity Theorem) *Let X be a finite-dimensional vector space and $T \in L(X, Y)$. Then $\operatorname{Im} T$ and $\ker T$ are vector subspaces of Y and X respectively, and*

$$\dim X = \dim \ker T + \operatorname{Rank} T$$

Proof: (Sketch) First show that $\operatorname{Im} T$ is a vector subspace of Y and $\ker T$ is a vector subspace of X (exercise).

Then let $V = \{v_1, \ldots, v_k\}$ be a basis for $\ker T$ (note that $\ker T \subseteq X$ so $\dim \ker T \leq \dim X = n$). If $\ker T = \{0\}$, take $k = 0$ so $V = \emptyset$. Extend V to a basis W for X with $W = \{v_1, \ldots, v_k, w_1, \ldots, w_r\}$. Then $\{T(w_1), \ldots, T(w_r)\}$ is a basis for $\operatorname{Im} T$ (do this as an exercise).

By definition, $\dim \ker T = k$ and $\dim \operatorname{Im} T = r$. Since W is a basis for X, $k + r = |W| = \dim X$, that is,

$$\dim X = \dim \ker T + \operatorname{Rank} T$$

■

Theorem 16 (Thm. 2.13) *$T \in L(X, Y)$ is one-to-one if and only if $\ker T = \{0\}$.*

Proof: Suppose T is one-to-one. Suppose $x \in \ker T$. Then $T(x) = 0$. But since T is linear, $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$. Since T is one-to-one, $x = 0$, so $\ker T = \{0\}$.

Conversely, suppose that $\ker T = \{0\}$. Suppose $T(x_1) = T(x_2)$. Then

$$\begin{aligned} T(x_1 - x_2) &= T(x_1) - T(x_2) \\ &= 0 \end{aligned}$$

which says $x_1 - x_2 \in \ker T$, so $x_1 - x_2 = 0$, or $x_1 = x_2$. Thus, T is one-to-one. ■

Definition 17 $T \in L(X, Y)$ is *invertible* if there is a function $S : Y \to X$ such that

$$\begin{aligned} S(T(x)) &= x \quad \forall x \in X \\ T(S(y)) &= y \quad \forall y \in Y \end{aligned}$$

In other words $S \circ T = id_X$ and $T \circ S = id_Y$, where id denotes the identity map. In this case denote S by T^{-1}.

Note that T is invertible if and only if it is one-to-one and onto. This is just the condition for the existence of an inverse *function.* The linearity of the inverse follows from the linearity of T.

Theorem 18 (Thm. 2.11) *If $T \in L(X, Y)$ is invertible, then $T^{-1} \in L(Y, X)$, i.e. T^{-1} is linear.*

Proof: Suppose $\alpha, \beta \in F$ and $v, w \in Y$. Since T is invertible, there exists unique $v', w' \in X$ such that

$$\begin{aligned} T(v') &= v \quad & T^{-1}(v) &= v' \\ T(w') &= w \quad & T^{-1}(w) &= w' \end{aligned} .$$

Then

$$\begin{aligned} T^{-1}(\alpha v + \beta w) &= T^{-1}\left(\alpha T(v') + \beta T(w')\right) \\ &= T^{-1}\left(T(\alpha v' + \beta w')\right) \\ &= \alpha v' + \beta w' \\ &= \alpha T^{-1}(v) + \beta T^{-1}(w) \end{aligned}$$

so $T^{-1} \in L(Y, X)$. ■

Theorem 19 (Thm. 3.2) *Let X, Y be two vector spaces over the same field F, and let $V = \{v_\lambda : \lambda \in \Lambda\}$ be a basis for X. Then a linear transformation $T \in L(X, Y)$ is completely determined by its values on V, that is:*

1. *Given any set $\{y_\lambda : \lambda \in \Lambda\} \subseteq Y$, $\exists T \in L(X, Y)$ s.t.*

$$T(v_\lambda) = y_\lambda \quad \forall \lambda \in \Lambda$$

2. *If $S, T \in L(X, Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$.*

Proof:

1. If $x \in X$, x has a unique representation of the form

$$x = \sum_{i=1}^{n} \alpha_i v_{\lambda_i} \text{ with } \alpha_i \neq 0 \ \forall i = 1, \ldots, n$$

(Recall that if $x = 0$, then $n = 0$.) Define

$$T(x) = \sum_{i=1}^{n} \alpha_i y_{\lambda_i}$$

Then $T(x) \in Y$. The verification that T is linear is left as an exercise.

2. Suppose $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$. Given $x \in X$,

$$\begin{aligned} S(x) &= S\left(\sum_{i=1}^{n} \alpha_i v_{\lambda_i}\right) \\ &= \sum_{i=1}^{n} \alpha_i S\left(v_{\lambda_i}\right) \\ &= \sum_{i=1}^{n} \alpha_i T\left(v_{\lambda_i}\right) \\ &= T\left(\sum_{i=1}^{n} \alpha_i v_{\lambda_i}\right) \\ &= T(x) \end{aligned}$$

so $S = T$.

■

Section 3.3. Isomorphisms

Definition 20 Two vector spaces X, Y over a field F are *isomorphic* if there is an invertible $T \in L(X, Y)$.

$T \in L(X, Y)$ is an *isomorphism* if it is invertible (one-to-one and onto).

Isomorphic vector spaces are essentially indistinguishable as vector spaces.

Theorem 21 (Thm. 3.3) *Two vector spaces* X, Y *over the same field are isomorphic if and only if* $\dim X = \dim Y$.

Proof: Suppose X, Y are isomorphic, and let $T \in L(X,Y)$ be an isomorphism. Let

$$U = \{u_\lambda : \lambda \in \Lambda\}$$

be a basis of X, and let

$$v_\lambda = T(u_\lambda),\ V = \{v_\lambda : \lambda \in \Lambda\}$$

Since T is one-to-one, U and V have the same cardinality. If $y \in Y$, then there exists $x \in X$ such that

$$\begin{aligned} y &= T(x) \\ &= T\left(\sum_{i=1}^{n} \alpha_{\lambda_i} u_{\lambda_i}\right) \\ &= \sum_{i=1}^{n} \alpha_{\lambda_i} T\left(u_{\lambda_i}\right) \\ &= \sum_{i=1}^{n} \alpha_{\lambda_i} v_{\lambda_i} \end{aligned}$$

which shows that V spans Y. To see that V is linearly independent, suppose

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \beta_i v_{\lambda_i} \\ &= \sum_{i=1}^{m} \beta_i T\left(u_{\lambda_i}\right) \\ &= T\left(\sum_{i=1}^{m} \beta_i u_{\lambda_i}\right) \end{aligned}$$

Since T is one-to-one, $\ker T = \{0\}$, so

$$\sum_{i=1}^{m} \beta_i u_{\lambda_i} = 0$$

Since U is a basis, we have $\beta_1 = \cdots = \beta_m = 0$, so V is linearly independent. Thus, V is a basis of Y; since U and V are numerically equivalent, $\dim X = \dim Y$.

Now suppose $\dim X = \dim Y$. Let

$$U = \{u_\lambda : \lambda \in \Lambda\} \text{ and } V = \{v_\lambda : \lambda \in \Lambda\}$$

be bases of X and Y; note we can use the same index set Λ for both because $\dim X = \dim Y$. By Theorem 3.2, there is a unique $T \in L(X,Y)$ such that $T(u_\lambda) = v_\lambda$ for all $\lambda \in \Lambda$. If $T(x) = 0$, then

$$\begin{aligned} 0 &= T(x) \\ &= T\left(\sum_{i=1}^{n} \alpha_i u_{\lambda_i}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\imath=1}^{n} \alpha_i T\left(u_{\lambda_i}\right) \\
&= \sum_{\imath=1}^{n} \alpha_i v_{\lambda_i} \\
&\Rightarrow \alpha_1 = \cdots = \alpha_n = 0 \text{ since } V \text{ is a basis} \\
&\Rightarrow x = 0 \\
&\Rightarrow \ker T = \{0\} \\
&\Rightarrow T \text{ is one-to-one}
\end{aligned}
$$

If $y \in Y$, write $y = \sum_{i=1}^{m} \beta_i v_{\lambda_i}$. Let

$$
x = \sum_{i=1}^{m} \beta_i u_{\lambda_i}
$$

Then

$$
\begin{aligned}
T(x) &= T\left(\sum_{i=1}^{m} \beta_i u_{\lambda_i}\right) \\
&= \sum_{i=1}^{m} \beta_i T(u_{\lambda_i}) \\
&= \sum_{i=1}^{m} \beta_i v_{\lambda_i} \\
&= y
\end{aligned}
$$

so T is onto, hence T is an isomorphism and X, Y are isomorphic. ■