Econ 204 – Problem Set 1[1]

Solutions

1. Use induction to prove the following:

   (a) For every $r \in \mathbb{N}$ and $x \in [-1, \infty)$, $(1+x)^r \geq 1 + rx$.

   (b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

Solution: (a) The case $x = -1$ is trivial, so we can focus on $x > -1$.

Base step: Let $r = 1$. Note that

$$(1+x)^r = 1 + x = 1 + rx$$

and so the formula is valid. Assume now that this is true for r. We then have that

$$\begin{aligned}
(1+x)^{r+1} &= (1+x)^r(1+x) \quad \text{by definition} \\
&\geq (1+rx)(1+x) \quad \text{by the induction step and } x > -1 \\
&= 1 + rx + x + rx^2 \quad \text{by the distributive property} \\
&= 1 + (r+1)x + rx^2 \\
&\geq 1 + (r+1)x \quad (rx^2 \geq 0)
\end{aligned}$$

(b) For $n = 1$, we have

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= 1 \\
&= \frac{1 * 2 * 3}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Assume now that this is true for n. We then have

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \quad \text{by definition} \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \text{by the induction step} \\
&= \frac{n+1}{6}(n(2n+1) + 6*(n+1)) \\
&= \frac{n+1}{6}(2n^2 + 7n + 6) \\
&= \frac{n+1}{6}(n+2)(2n+3) \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}
\end{aligned}$$

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2. Prove the following statements:
   (a) Let X an infinite set. Prove that there exists $A \subseteq X$ such that A is countable.
   (b) Show that if X is an infinite set, then there is an injection $r : \mathbb{N} \to X$. (Recall from lecture 2 this implies $|\mathbb{N}| \leq |X|$, thus the cardinality of the natural numbers N is less than or equal to the cardinality of any infinite set.)

Solution:

(a) We will proceed by induction showing that for all $n \in \mathbb{N}$ there exists a subset of X with n distinct elements.

Case $n = 1$: Since X is infinite, there exists $x \in X$, denote this by x_1.

Suppose now that we have found $x_1, \ldots, x_n$ such that $x_i \neq x_j$ for $i \neq j$. Since X is infinite, there exists $y \in X \setminus \{x_1, \ldots, x_n\}$, let $x_{n+1} = y$. We then have that $\{x_1, \ldots, x_{n+1}\}$ is a subset of X with $n + 1$ distinct elements.

Let $A_n = \{x_1, \ldots, x_n\}$ and $A = \bigcup_{n=1}^{\infty} A_n$. Since $A_n \subseteq X$ for all n, $A \subseteq X$. Let $f : \mathbb{N} \to A$ be defined as $f(n) = x_n$. By construction, f is bijective, and thus A is countable.

(b) Let X be an arbitrary infinite set. By (a), we know that there exists $A \subset X$ such that A is countable. Define $f : A \to X$ as $f(a) = a$, which is well defined and injective by construction. Let $g : \mathbb{N} \to A$ be a bijection, which exists because A is countable. The composition $f \circ g : \mathbb{N} \to X$ is an injective function, which proves the result.

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3. Let A, B be sets. Show that
   (a) $A \subseteq B \iff A \cap B^C = \varnothing$
   (b) $A = B \iff (A \cap B^C) \cup (A^C \cap B) = \varnothing$
   (c) A function $f : A \to B$ is injective iff $\forall X \subseteq A \; f(A \setminus X) = f(A) \setminus f(X)$

Solution:

(a) We will first show $\Rightarrow$. Suppose that $A \subseteq B$. Then

$$\begin{aligned} x \in A &\Rightarrow x \in B \\ &\Rightarrow x \notin B^C \end{aligned}$$

Thus, for any $x \in A$ we have that $x \notin B^C$, and $A \cap B^C = \varnothing$

We now show $\Leftarrow$.

Suppose $A \cap B^C = \varnothing$. Suppose, by contradiction, that $\exists a \in A$ such that $a \notin B$. Then $a \in A$ and $a \in B^C$, and thus $a \in A \cap B^C$, a contradiction

(b) We will do both implications for this one at once

$$\begin{aligned} A = B &\iff A \subseteq B \wedge B \subseteq A \\ &\iff (A \cap B^C = \varnothing) \wedge (A^C \cap B = \varnothing) \quad \text{(by 1.a)} \\ &\iff (A \cap B^C) \cup (A^C \cap B) = \varnothing \end{aligned}$$

(c) Let's assume that f is injective and let $X \subseteq A$ be arbitrary. Notice that

$$\begin{aligned} y \in f(A \setminus X) &\iff \exists x \in A \setminus X \ f(x) = y \\ &\iff \exists x \in A \ f(x) = y \wedge \forall z \in X \ f(z) \neq y \quad \text{(f is injective)} \\ &\iff y \in f(A) \wedge y \notin f(X) \\ &\iff y \in f(A) \setminus f(X) \end{aligned}$$

Now let's assume that $\forall X \subseteq A \, f(A \setminus X) = f(A) \setminus f(X)$ and let's suppose, by contradiction, that f is not injective.

Since f is not injective, there exist $x, z \in A$ such that $x \neq z$ and $f(x) = f(z) = y$. Let $X = \{x\}$. Since $z \neq x$, we have that $z \in A \setminus X$, and thus

$$y \in f(A \setminus X)$$

But since $y = f(x)$, we have that $y \in f(X)$, and thus $y \notin f(A) \setminus f(X)$. Thus $f(A \setminus X) \neq f(A) \setminus f(X)$, which contradicts our initial assumption.

4. In this exercise we will practice working with sets whose elements are sets as well. For this, we will need the following definition:

   Sigma-Algebra: Let Ω be a set and $\mathcal{F} \subseteq 2^\Omega$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a sigma-algebra if the following properties hold:

   - $\Omega \in \mathcal{F}$

- If $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$.
- If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of sets such that $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}$, then $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

(a) Prove that if $\mathcal{F}$ is a sigma-algebra and $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(b) Prove that if $\mathcal{F}$ is a sigma-algebra, then $\varnothing \in \mathcal{F}$

(c) Prove that $\{\varnothing, \Omega\}$ is a sigma-algebra. Argue that this is the smallest sigma-algebra over the set Ω.

(d) Prove that 2^Ω is a sigma-algebra. Argue that this is the largest sigma-algebra over the set Ω.

(e) Prove that if $\mathcal{F}_1, \mathcal{F}_2$ are sigma-algebras, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a sigma-algebra.

(f) Prove that if $\{\mathcal{F}_a\}_{a\in\mathcal{A}}$ is a collection of sigma-algebras, then $\cap_{a\in\mathcal{A}} \mathcal{F}_a$ is a sigma-algebra. (Note that we have made no restriction on the set $\mathcal{A}$.)

(g) Prove or provide a counterexample to the following statement: If $\mathcal{F}_1, \mathcal{F}_2$ are sigma-algebras, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a sigma-algebra.

(h) Let $\Omega = \{1, 2, 3\}$. List all the possible sigma-algebras over Ω. (There are surprisingly few).

Solution:

(a) Since $A, B \in \mathcal{F}$, we have that $A^C, B^C \in \mathcal{F}$. Since $A^C, B^C \in \mathcal{F}$, we have that $A^C \cup B^C \in \mathcal{F}$, and thus $(A^C \cup B^C)^C \in \mathcal{F}$. By DeMorgan's Law, $A \cap B = (A^C \cup B^C)^C \in \mathcal{F}$.

(b) Since $\mathcal{F}$ is a sigma-algebra, we know that $\Omega \in \mathcal{F}$. Since $\Omega \in \mathcal{F}$, we have that $\varnothing = \Omega^C \in \mathcal{F}$.

(c) We automatically have that $\Omega \in \{\varnothing, \Omega\}$, so the first property holds.

Let now A be an element of $\{\varnothing, \Omega\}$. If $A = \varnothing$, then $A^C = \Omega \in \{\varnothing, \Omega\}$. If $A = \Omega$, then $A^C = \varnothing \in \{\varnothing, \Omega\}$, and thus the second property holds.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of sets such that $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}$.

Case 1: $\forall n \in \mathbb{N}\ A_n = \varnothing$. Then $\cup_{n\in\mathbb{N}} A_n = \varnothing \in \mathcal{F}$.
Case 2: $\exists n \in \mathbb{N}\ A_n = \Omega$. Then $\cup_{n\in\mathbb{N}} A_n = \Omega \in \mathcal{F}$.

This is the smallest sigma-algebra over Ω since we have already shown that if $\mathcal{F}$ is any other sigma-algebra, $\{\varnothing, \Omega\} \subseteq \mathcal{F}$

(d) $\Omega \in 2^\Omega$ trivially. If $A \subseteq \Omega$, then $A^C = \Omega \setminus A \subseteq \Omega$, and thus $A^C \in 2^\Omega$. Lastly, if $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of sets such that $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}$, then

$$\forall n \in \mathbb{N}\ A_n \subseteq \Omega \Rightarrow \cup_{n\in\mathbb{N}} A_n \subseteq \Omega \Rightarrow \cup_{n\in\mathbb{N}} A_n \in 2^\Omega$$

Since $\mathcal{F} \subseteq 2^\Omega$ for any sigma-algebra $\mathcal{F}$, 2^Ω is the largest possible sigma-algebra.

(e) Let $\mathcal{F}_1, \mathcal{F}_2$ be sigma-algebras.
Since $\mathcal{F}_1, \mathcal{F}_2$ are sigma-algebras, $\Omega \in \mathcal{F}_1$ and $\Omega \in \mathcal{F}_2$, and thus $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$.

Let $A \in \mathcal{F}_1 \cap \mathcal{F}_2$. Since,$A \in \mathcal{F}_1$, $A^C \in \mathcal{F}_1$. Since,$A \in \mathcal{F}_2$, $A^C \in \mathcal{F}_2$. Thus, $A^C \in \mathcal{F}_1 \cap \mathcal{F}_2$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of sets such that $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$.

Since $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}_1$, we have that $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_1$. Since $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}_2$, we have that $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_2$. Thus $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$.

(f) Let $\{\mathcal{F}_a\}_{a\in\mathcal{A}}$ be a collection of sigma-algebras.
Since $\{\mathcal{F}_a\}_{a\in\mathcal{A}}$ are sigma-algebras, $\forall a \in \mathcal{A}\ \Omega \in \mathcal{F}_a$, and thus $\Omega \in \cap_{a\in\mathcal{A}} \mathcal{F}_a$.

Let $A \in \cap_{a\in\mathcal{A}} \mathcal{F}_a$. Since,$\forall a \in \mathcal{A}\ A \in \mathcal{F}_a$, $A^C \in \mathcal{F}_a \forall a \in \mathcal{A}$. Thus, $A^C \in \cap_{a\in\mathcal{A}} \mathcal{F}_a$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of sets such that $\forall n \in \mathbb{N}\ A_n \in \cap_{a\in\mathcal{A}} \mathcal{F}_a$.

Let $a \in \mathcal{A}$ be arbitrary. Since $\forall n \in \mathbb{N}\ A_n \in \mathcal{F}_a$, we have that $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_a$. Thus $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_a \forall a \in \mathcal{A}$, and $\cup_{n\in\mathbb{N}} A_n \in \cap_{a\in\mathcal{A}} \mathcal{F}_a$.

(g) Let $\Omega = \{1, 2, 3\}$, $\mathcal{F}_1 = \{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{F}_2 = \{\varnothing, \{2\}, \{1, 3\}, \{1, 2, 3\}\}$. Checking that these are sigma-algebras is part of the next question.
Note that $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\varnothing, \{1\}, \{2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. Note that $\{2, 3\} \cap \{1, 3\} = \{3\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$. By the first exercise of this problem, this implies that $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a sigma-algebra.

(h) The possible sigma algebras are:

i. $\{\varnothing, \{1, 2, 3\}\}$,
ii. $\{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$
iii. $\{\varnothing, \{2\}, \{1, 3\}, \{1, 2, 3\}\}$

iv. $\{\varnothing, \{3\}, \{1,2\}, \{1,2,3\}\}$

v. $\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

The reasoning is as follows. If a sigma-algebra does not contain a singleton, then it must not contain any set with two elements, since sigma-algebras are closed by taking complements. In this case, the only sigma-algebra possible is i).

If a sigma-algebra contains one singleton, there are two cases. If it contains only one singleton, it must also contain its complement, and these are cases ii-iv. If it contains two singletons, it must contain their union, and since the complement of their union is the remaining singleton, we must be on the last case.

5. In this exercise we will practice working with unions and intersections of sets. Let Ω be a set $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of subsets of Ω. Define:

$$\limsup(A_n) = \bigcap_{m\geq 1}\bigcup_{k\geq m} A_k$$
$$\liminf(A_n) = \bigcup_{m\geq 1}\bigcap_{k\geq m} A_k$$

(a) Show that:

$$\limsup(A_n) = \{x \in \Omega \mid \forall m \in \mathbb{N} \exists k \geq m \in \mathbb{N}\ x \in A_k\}$$
$$\liminf(A_n) = \{x \in \Omega \mid \exists m \in \mathbb{N} \forall k \geq m \in \mathbb{N}\ x \in A_k\}$$

Argue that $\limsup(A_n)$ is the set of points that appear infinitely often in the sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, and $\liminf(A_n)$ is the set of points that are "eventually" in the sequence of sets $\{A_n\}_{n\in\mathbb{N}}$. (You don't have to argue this formally, I just want you to practice developing an intuitive understanding for the definition of sets using symbols).

(b) Show that $\liminf(A_n) \subseteq \limsup(A_n)$

(c) Find an example of $\{A_n\}_{n\in\mathbb{N}}$ such that $\limsup(A_n) \not\subseteq \liminf(A_n)$

(d) Find an example of $\{A_n\}_{n\in\mathbb{N}}$ such that $\forall k \in \mathbb{N}\ A_k \subset \limsup(A_n)$ and $\liminf(A_n) = \varnothing$

(e) Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is such that $\forall n \in \mathbb{N}\ A_n \subseteq A_{n+1}$. Prove that $\liminf(A_n) = \limsup(A_n)$

(f) Show that $\liminf(A_n) = (\limsup(A_n^C))^C$

(g) Let $\mathcal{F}$ be a sigma-algebra and $\{A_n\}_{n\in\mathbb{N}}$ be such that $\forall n \in \mathbb{N} A_n \in \mathcal{F}$. Show that $\liminf(A_n), \limsup(A_n) \in \mathcal{F}$. (See Problem 4 for the definition of a sigma-algebra.)

Solution:

(a) Note that

$$\begin{aligned} x \in \liminf(A_n) &\iff x \in \bigcup_{n\geq 1}\bigcap_{k\geq n} A_k \\ &\iff \exists n \in \mathbb{N}\ x \in \bigcap_{k\geq n} A_k \\ &\iff \exists n \in \mathbb{N} \forall k \geq n\ x \in A_k \\ x \in \limsup(A_n) &\iff x \in \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k \\ &\iff \forall n \in \mathbb{N}\ x \in \bigcup_{k\geq n} A_k \\ &\iff \forall n \in \mathbb{N} \exists k \geq n\ x \in A_k \end{aligned}$$

Given $x \in \limsup(A_n)$ and any $n \in \mathbb{N}$, we can find index $k \geq n$ such that $x \in A_k$. By repeating this argument for larger and larger indices, we see that it must be the case that $x \in A_m$ for infinitely many indices m. Conversely, if it is not true that $x \in A_m$ for infinitely many indices m, then there must exist $N \in \mathbb{N}$ such that $\forall n > N\ x \notin A_n$. This implies that $x \notin \bigcup_{k\geq N} A_k$, and thus $x \notin \limsup(A_n)$.

If $x \in \liminf(A_n)$, there exists $N \in \mathbb{N}$ such that $x \in \bigcap_{k\geq N} A_k$. This implies that $\forall k \geq N\ x \in A_k$, and thus x is eventually in all sets A_n. Conversely, if x is not eventually in all sets A_n, for any $N \in \mathbb{N}$ we can find $k \geq n$ such that $x \notin A_k$. This implies that $x \notin \bigcap_{k\geq N} A_k$. Since N was taken arbitrary, $x \notin \bigcup_{n\geq 1}\bigcap_{k\geq n} A_k = \liminf(A_n)$.

(b) Let $x \in \liminf(A_n)$. By definition, there exists $n_x \in \mathbb{N}$ such that $\forall k \geq n_x\ x \in A_k$.
Let now $n \in \mathbb{N}$ be arbitrary. Notice that for $k = \max\{n, n_x\}$, we have that $k \geq n_x$, and thus $x \in A_k$. Since $k \geq n$, we also have that $x \in \bigcup_{k\geq n} A_k$. Since n was arbitrary, we have that $\forall n \in \mathbb{N}\ x \in \bigcup_{k\geq n} A_k$, and thus $x \in \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k = \limsup(A_n)$.

(c) Let

$$A_n = \begin{cases} \{0\}, & \text{if n is odd} \\ \{1\}, & \text{if n is even} \end{cases}$$

Then $\liminf(A_n) = \varnothing \subseteq \{0, 1\} = \limsup(A_n)$.

(d) See the example in c).

(e) By b), we just need to show that $\limsup(A_n) \subseteq \liminf(A_n)$.

Let $x \in \limsup(A_n)$ and $n \in \mathbb{N}$ be arbitrary. Since $x \in \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k$, we know that $x \in \bigcup_{k\geq n} A_k$, and thus there is $k_x \geq n$ such that $x \in A_{k_x}$. We then have that $\forall k \geq k_x$ $A_{k_x} \subseteq A_k$, and thus $x \in \bigcap_{k\geq k_x} A_k$. This implies that $x \in \bigcup_{n\geq 1}\bigcap_{k\geq n} A_k = \liminf(A_n)$, which concludes the proof.

(f) Note that

$$\begin{aligned}
x \in \liminf(A_n) &\iff \exists n \in \mathbb{N} \forall k \geq n \; x \in A_k \\
&\iff \exists n \in \mathbb{N} \forall k \geq n \; \neg(x \in A_k^C) \\
&\iff \exists n \in \mathbb{N} \; \nexists k \geq n \; x \in A_k^C \\
&\iff \neg \forall n \in \mathbb{N} \exists k \geq n \; x \in A_k^C \\
&\iff \neg x \in \limsup(A_n^C) \\
&\iff x \in (\limsup(A_n^C))^C
\end{aligned}$$

(g) Since $\forall n \in \mathbb{N}$ $A_n \in \mathcal{F}$, we have that $\forall n \in \mathbb{N}$ $\bigcup_{k\geq n} A_k \in \mathcal{F}$. Using an argument analogous to the one in 4.a), we can show that sigma-algebras are closed by countable intersections. This implies that $\limsup(A_n) = \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k \in \mathcal{F}$.

For the lim inf, this follows directly from sigma-algebras being closed by taking complements and the result we just proved.

6. Let $X \subseteq \mathbb{R}$. We say that a function $f : X \to \mathbb{R}$ is bounded if its image $f(X) \subseteq \mathbb{R}$ is a bounded set. We then write $\sup_f = \sup f(X)$ and $\inf_f = \inf f(X)$. Show that

   (a) If $f, g : X \to \mathbb{R}$ are bounded, $f + g : X \to \mathbb{R}$ is bounded

   (b) Show that $(f + g)(X) \subset f(X) + g(X)$ and provide a counterexample in which the inclusion is strict. [2]

   (c) Show that $\sup_{f+g} \leq \sup_f + \sup_g$ and $\inf_{f+g} \geq \inf_f + \inf_g$

   (d) Provide an example for which the inequalities in the previous item are strict.

   (e) Show that $f \cdot g : X \to \mathbb{R}$ is bounded

   (f) Show that $(f \cdot g)(X) \subset f(X) \cdot g(X)$ [3]

[2] Given $A, B \subseteq \mathbb{R}$ non-empty and bounded, we define $A + B = \{z \in \mathbb{R} | z = x + y, x \in A, y \in B\}$

[3] Given $A, B \subseteq \mathbb{R}$ non-empty and bounded, we define $A \cdot B = \{z \in \mathbb{R} | z = x \cdot y, x \in A, y \in B\}$.

(g) Show that, if f and g are both positive, then $\sup_{f\cdot g} \leq \sup_f \cdot \sup_g$ and $\inf_{f\cdot g} \geq \inf_f \cdot \inf_g$

(h) Provide an example for which the inequalities in the previous item are strict.

(i) Provide a counterexample for item g) if the functions are not positive.

(j) Show that if f is positive, $\sup_{f^2} = (\sup_f)^2$

Solution:

(a) Since f and g are bounded, there exist $a_f, b_f, a_g, b_g \in \mathbb{R}$ such that

$$\forall x \in X \; a_f \leq f(x) \leq b_f$$
$$\forall x \in X \; a_g \leq g(x) \leq b_g$$

Note then that

$$\begin{aligned}\forall x \in X \; (f+g)(x) &= f(x) + g(x) \\ &\leq b_f + b_g\end{aligned}$$

and

$$\begin{aligned}\forall x \in X \; (f+g)(x) &= f(x) + g(x) \\ &\geq a_f + a_g\end{aligned}$$

and thus $(f+g)(X)$ is bounded, and $(f+g)$ is bounded.

(b) Let $y \in (f+g)(X)$. We know that $\exists x \in X \; y = (f+g)(x)$. Let $y_1 = f(x) \in f(X), y_2 = g(x) \in g(X)$. We then have that $y = y_1 + y_2$, and $y \in f(X) + g(X)$.

For the counterexample, let $X = [-1, 1], f(x) = x, g(x) = -x$. Notice that,

$$\forall x \in X (f+g)(x) = f(x) + g(x) = x + (-x) = 0$$

and thus $(f+g)(X) = \{0\}$. Now, note that $f(1) = g(-1) = 1$, and thus $2 \in f(X) + g(X)$. Thus the inclusion is strict.

(c) Note that

$$\forall x \in X \; (f+g)(x) = f(x) + g(x) \leq \sup_f + \sup_g \Rightarrow \sup_{f+g} \leq \sup_f + \sup_g$$

Similarly,

$$\forall x \in X \; (f+g)(x) = f(x) + g(x) \geq \inf_f + \inf_g \Rightarrow \inf_{f+g} \leq \inf_f + \inf_g$$

(d) Again let $X = [-1, 1], f(x) = x, g(x) = -x$. Notice that

$$\begin{aligned} \sup_{f+g} &= 0 < 1 + 1 = \sup_f + \sup_g \\ \inf_{f+g} &= 0 > -1 - 1 = \inf_f + \inf_g \end{aligned}$$

(e) Since f and g are bounded, there exist $a_f, b_f, a_g, b_g \in \mathbb{R}$ such that

$$\begin{aligned} \forall x \in X \; a_f \leq f(x) \leq b_f \\ \forall x \in X \; a_g \leq g(x) \leq b_g \end{aligned}$$

If $f(x) \geq 0$, then $|f(x)| \leq |b_f|$, and if $f(x) \leq 0$, $|f(x)| \leq |a_f|$. Thus $|f(x)| \leq |a_f| + |b_f|$, and $|f|$ is bounded. Similarly, $|g|$ is bounded. Thus

$$(f \cdot g)(x) = f(x) \cdot g(x) \leq |f(x)| \cdot |g(x)| \leq \sup_{|f|} \sup_{|g|}$$

The lower bound can be obtained analogously.

(f) Note that

$$\begin{aligned} y \in (f \cdot g)(X) &\Rightarrow \exists x \in X \; y = (f \cdot g)(x) \\ &\Rightarrow \exists x \in X \; y = f(x) \cdot g(x) \\ &\Rightarrow y = y_1 \cdot y_2, \; y_1 = f(x) \in f(X), y_2 = g(x) \in g(X) \\ &\Rightarrow y \in f(X) \cdot g(X) \end{aligned}$$

(g) Let $x \in X$ be arbitrary. First note that, since $f \geq 0$, 0 is a lower bound to f, and by definition

$$0 \leq \inf_f \leq f(x) \leq \sup_f \; \forall x \in X$$

And similarly for g:

$$0 \leq \inf_g \leq g(x) \leq \sup_g \ \forall x \in X$$

Remember now that, for $a, b, c, d \in \mathbb{R}$

$$0 < a < b \wedge 0 < c < d \Rightarrow 0 < ac < bd$$

Combining the statements above, we have that:

$$(f \cdot g)(x) = f(x) \cdot g(x) \leq \sup_f \sup_g$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \geq \inf_f \inf_g$$

(h) Let $X = [1/2, 2], f(x) = x, g(x) = 1/x$. Notice that

$$\sup_{f \cdot g} = 1 < 2 \cdot 2 = \sup_f \cdot \sup_g$$
$$\inf_{f \cdot g} = 1 > 1/4 = 1/2 \cdot 1/2 = \inf_f \cdot \inf_g$$

(i) Let $X = [0, 1], f(x) = g(x) = -x$. Note that

$$\sup_f = \sup_g = 0$$

But $(f \cdot g)(x) = x^2$, and thus

$$\sup_{f \cdot g} = 1$$

Similarly,

$$\inf_f = \inf_g = -1$$

But $(f \cdot g)(x) = x^2$, and thus

$$\inf_{f\cdot g} = 0$$

(j) We already know that $\sup_{f^2} \leq (\sup_f)^2$. So we just need to prove the other inequality.

Since f is bounded, f^2 is bounded, and since f and f^2 are both positive, $\sup f$ and $\sup f^2$ are both nonnegative real numbers.

If $\sup f = 0$, then $f(x) = 0$ for all x, which implies $(\sup f)^2 = \sup f^2 = 0$. So suppose $\sup f > 0$. Then consider $y \in \mathbb{R}$ such that

$$0 \leq y < (\sup f)^2$$

This implies that $0 \leq \sqrt{y} < \sup f$. By the definition of sup, there exists $x \in X$ such that $f(x) > \sqrt{y}$, which implies $f^2(x) > y$. Then

$$\sup f^2 \geq f^2(x) > y$$

Since this holds for every $y < (\sup f)^2$, $\sup f^2 \geq (\sup f)^2$.