Econ 204 – Problem Set 1[1]

Due Friday July 28, 2023 11:59PM

1. Use induction to prove the following:

   (a) For every $r \in \mathbb{N}$ and $x \in [-1, \infty)$, $(1+x)^r \geq 1 + rx$.

   (b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

2. Prove the following statements:

   (a) Let X an infinite set. Prove that there exists $A \subseteq X$ such that A is countable.

   (b) Show that if X is an infinite set, then there is an injection $r : \mathbb{N} \to X$. (Recall from lecture 2 this implies $|\mathbb{N}| \leq |X|$, thus the cardinality of the natural numbers N is less than or equal to the cardinality of any infinite set.)

3. Let A, B be sets. Show that

   (a) $A \subseteq B \iff A \cap B^C = \varnothing$

   (b) $A = B \iff (A \cap B^C) \cup (A^C \cap B) = \varnothing$

   (c) A function $f : A \to B$ is injective iff $\forall X \subseteq A \ f(A \setminus X) = f(A) \setminus f(X)$

4. In this exercise we will practice working with sets whose elements are sets as well. For this, we will need the following definition:

   Sigma-Algebra: Let Ω be a set and $\mathcal{F} \subseteq 2^{\Omega}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a sigma-algebra if the following properties hold:

   - $\Omega \in \mathcal{F}$
   - If $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$.
   - If $\{A_n\}_{n \in \mathbb{N}}$ is a countable collection of sets such that $\forall n \in \mathbb{N} \ A_n \in \mathcal{F}$, then $\cup_{n \in \mathbb{N}} A_n \in \mathcal{F}$.

   (a) Prove that if $\mathcal{F}$ is a sigma-algebra and $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

   (b) Prove that if $\mathcal{F}$ is a sigma-algebra, then $\varnothing \in \mathcal{F}$

   (c) Prove that $\{\varnothing, \Omega\}$ is a sigma-algebra. Argue that this is the smallest sigma-algebra over the set Ω.

   (d) Prove that 2^{Ω} is a sigma-algebra. Argue that this is the largest sigma-algebra over the set Ω.

   (e) Prove that if $\mathcal{F}_1, \mathcal{F}_2$ are sigma-algebras, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a sigma-algebra.

   (f) Prove that if $\{\mathcal{F}_a\}_{a \in \mathcal{A}}$ is a collection of sigma-algebras, then $\cap_{a \in \mathcal{A}} \mathcal{F}_a$ is a sigma-algebra. (Note that we have made no restriction on the set $\mathcal{A}$.)

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

(g) Prove or provide a counterexample to the following statement: If $\mathcal{F}_1, \mathcal{F}_2$ are sigma-algebras, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a sigma-algebra.

(h) Let $\Omega = \{1, 2, 3\}$. List all the possible sigma-algebras over Ω. (There are surprisingly few).

5. In this exercise we will practice working with unions and intersections of sets. Let Ω be a set $\{A_n\}_{n \in \mathbb{N}}$ be a countable collection of subsets of Ω. Define:

$$\limsup(A_n) = \bigcap_{m \geq 1} \bigcup_{k \geq m} A_k$$
$$\liminf(A_n) = \bigcup_{m \geq 1} \bigcap_{k \geq m} A_k$$

(a) Show that:

$$\limsup(A_n) = \{x \in \Omega \mid \forall m \in \mathbb{N} \exists k \geq m \in \mathbb{N}\ x \in A_k\}$$
$$\liminf(A_n) = \{x \in \Omega \mid \exists m \in \mathbb{N} \forall k \geq m \in \mathbb{N}\ x \in A_k\}$$

Argue that $\limsup(A_n)$ is the set of points that appear infinitely often in the sequence of sets $\{A_n\}_{n \in \mathbb{N}}$, and $\liminf(A_n)$ is the set of points that are "eventually" in the sequence of sets $\{A_n\}_{n \in \mathbb{N}}$. (You don't have to argue this formally, I just want you to practice developing an intuitive understanding for the definition of sets using symbols).

(b) Show that $\liminf(A_n) \subseteq \limsup(A_n)$

(c) Find an example of $\{A_n\}_{n \in \mathbb{N}}$ such that $\limsup(A_n) \not\subseteq \liminf(A_n)$

(d) Find an example of $\{A_n\}_{n \in \mathbb{N}}$ such that $\forall k \in \mathbb{N}\ A_k \subset \limsup(A_n)$ and $\liminf(A_n) = \varnothing$

(e) Suppose that $\{A_n\}_{n \in \mathbb{N}}$ is such that $\forall n \in \mathbb{N}\ A_n \subseteq A_{n+1}$. Prove that $\liminf(A_n) = \limsup(A_n)$

(f) Show that $\liminf(A_n) = (\limsup(A_n^C))^C$

(g) Let $\mathcal{F}$ be a sigma-algebra and $\{A_n\}_{n \in \mathbb{N}}$ be such that $\forall n \in \mathbb{N} A_n \in \mathcal{F}$. Show that $\liminf(A_n), \limsup(A_n) \in \mathcal{F}$. (See Problem 4 for the definition of a sigma-algebra.)

6. Let $X \subseteq \mathbb{R}$. We say that a function $f : X \to \mathbb{R}$ is bounded if its image $f(X) \subseteq \mathbb{R}$ is a bounded set. We then write $\sup_f = \sup f(X)$ and $\inf_f = \inf f(X)$. Show that

(a) If $f, g : X \to \mathbb{R}$ are bounded, $f + g : X \to \mathbb{R}$ is bounded

(b) Show that $(f+g)(X) \subset f(X)+g(X)$ and provide a counterexample in which the inclusion is strict. [2]

(c) Show that $\sup_{f+g} \leq \sup_f + \sup_g$ and $\inf_{f+g} \geq \inf_f + \inf_g$

(d) Provide an example for which the inequalities in the previous item are strict.

(e) Show that $f \cdot g : X \to \mathbb{R}$ is bounded

(f) Show that $(f \cdot g)(X) \subset f(X) \cdot g(X)$ [3]

(g) Show that, if f and g are both positive, then $\sup_{f \cdot g} \leq \sup_f \cdot \sup_g$ and $\inf_{f \cdot g} \geq \inf_f \cdot \inf_g$

(h) Provide an example for which the inequalities in the previous item are strict.

(i) Provide a counterexample for item g) if the functions are not positive.

(j) Show that if f is positive, $\sup_{f^2} = (\sup_f)^2$

[2] Given $A, B \subseteq \mathbb{R}$ non-empty and bounded, we define $A + B = \{z \in \mathbb{R} | z = x + y, x \in A, y \in B\}$

[3] Given $A, B \subseteq \mathbb{R}$ non-empty and bounded, we define $A \cdot B = \{z \in \mathbb{R} | z = x \cdot y, x \in A, y \in B\}$.