Econ 204 – Problem Set 2[1]

Solutions

1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

   Solution:

   Define the mapping $f : \mathbb{R} \to (-1, 1)$ as

   $$f(x) = \frac{x}{1 + |x|} \tag{1}$$

   f is a continuous bijection (note $f(0) = 0$), where the inverse $f^{-1} : (-1, 1) \to \mathbb{R}$ is

   $$f^{-1}(y) = \frac{y}{1 - |y|} \tag{2}$$

   which is also continuous at all $-1 < y < 1$ (again note $f^{-1}(0) = 0$). Thus f is a homeomorphism. With the usual metric, $\mathbb{R}$ is complete but $(-1, 1)$ is incomplete.

2. Given $A, B \subseteq \mathbb{R}^n$, we define the sum of these two sets by:

   $$A + B = \{a + b \mid a \in A, b \in B\}$$

   Prove or find a counterexample to the following statements:

   (a) If either A or B is an open set, then $A + B$ is an open set.

   (b) If both A and B are closed sets, $A + B$ is a closed set.

   Solution:

   (a) This is true. Without loss of generality, assume that A is open.
   Let $c = a + b$ for $a \in A, b \in B$. Since A is open and $a \in A$, we know that there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$. Let now $x \in B(c, \varepsilon)$ be arbitrary, and take $y = x - b$. Note that

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

$$
\begin{aligned}
||a-y|| &= ||a-(x-b)|| \\
&= ||a+b-x|| \\
&= ||c-x|| \\
&\leq \varepsilon
\end{aligned}
$$

And thus $y \in B(a,\varepsilon) \subseteq A$. Since $x = x - b + b = y + b$. This implies that $x \in A + C$, and thus $A + C$ is open.

(b) This is not true. Let $n = 1$ and $A = \{n \mid n \in \mathbb{N}\}, B = \{-n + 1/n \mid n \in \mathbb{N}\}$. Note that both sets are closed. However, we have that $\{1/n\}_{n\in\mathbb{N}} \subseteq A + B$, $0 \notin A + B$, but $\forall \varepsilon > 0 \exists n \in \mathbb{N}\ |1/n - 0| = |1/n| < \varepsilon$, and thus $A + B$ is not closed.

3. Show that

(a) If $A \subseteq \mathbb{R}$ is open and $a_1, \ldots, a_n \in A$, then $A \setminus \{a_1, \ldots, a_n\}$ is open.

(b) If A is open and $A \cap \overline{B} \neq \varnothing$, then $A \cap B \neq \varnothing$

(c) If $\mathcal{A}$ is a collection of open subsets of $\mathbb{R}^n$, pairwise disjoint, that is, $A_\lambda \cap A_{\lambda'} = \varnothing$ if $\lambda \neq \lambda'$, then $\mathcal{A}$ is at most countable

(d) The set of limit points of any set $A \subseteq \mathbb{R}^n$ is closed.

Solution:

(a) First we know that $\{a\}$ is a closed set for any point, and the finite union of closed sets is closed, thus $\{a_1, \ldots, a_n\}$ is always closed. Thus $\{a_1, \ldots, a_n\}^C$ is open, and since the finite intersection of open sets is open, $A \cap \{a_1, \ldots, a_n\}^C = A \setminus \{a_1, \ldots, a_n\}$ is open.

(b) Since $A \cap \overline{B} \neq \varnothing$, there exists $a \in A$ and $a \in \overline{B}$

Since $a \in A$ and A is an open set, we have that $\exists \varepsilon_a > 0$ such that $B(a, \varepsilon_a) \subseteq A$.

Since $a \in \overline{B}$, $\forall \varepsilon > 0$ we have that $B(a,\varepsilon) \cap B \neq \varnothing$. Particularly, there exists $b \in B(a, \varepsilon_a) \cap B$. But since $B(a, \varepsilon_a) \subseteq A$, this implies that $b \in A$, and thus $b \in A \cap B$.

(c) We will first prove that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be open and non-empty.

Since A is non-empty, there exists $a = (a_1, \ldots, a_n) \in A$. Since A is open, there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$.

Let $b = \frac{\varepsilon}{2\sqrt{n}}$, and note that $a + b\mathbf{1} = (a_1 + b, \ldots, a_n + b) \in B(a, \varepsilon)$, since

$$\|a + b\mathbf{1} - a\| = b\|\mathbf{1}\| = \frac{\varepsilon}{2\sqrt{n}}\sqrt{n} = \frac{\varepsilon}{2} \tag{3}$$

Then for each i, there exists $q_i \in \mathbb{Q}$ such that $q_i \in (a_i, a_i + b)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Let $q = (q_1, \ldots, q_n)$. Then $q \in \mathbb{Q}^n$ and

$$\begin{aligned} \|a - q\|^2 &= \|(a_1 - q_1, \ldots, a_n - q_n)\|^2 \\ &= \sum_{i=1}^{n} (a_i - q_i)^2 \\ &\leq \sum_{i=1}^{n} b^2 \\ &= nb^2 < \varepsilon^2 \end{aligned}$$

So $\|a - q\| < \varepsilon$. Thus $q \in B(a, \varepsilon) \subseteq A$, which shows that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Now we can go back to the original question. Suppose, by contradiction, that $\mathcal{A}$ is uncountable. For each $A \in \mathcal{A}$, we can take $q_a \in A \cap \mathbb{Q}^n$.

Since all elements in $\mathcal{A}$ are disjoint, this implies that $q_a \neq q_{a'}$ if $A \neq A'$. This implies that $\{q_a\}_{a \in \mathcal{A}}$ is an uncountable subset of $\mathbb{Q}^n$, which is impossible, since it is a finite product of countable sets, and thus countable.

(d) Suppose this is not the case. Let B be the set of limit points of A. Then there exists a sequence $\{b_n\}_{n \in \mathbb{N}} \subseteq B$ such that $b_n \to b \notin B$. Moreover, taking a subsequence if necessary, take b_n such that $||b_n - b|| < 1/4n$.

Now let, for each $n \in \mathbb{N}$, $a_n \in A$ such that $a_n \in B(b_n, 1/4n)$, which exist since b_n is a limit point of A for each n. Note that

$$||b - a_n|| \leq ||b - b_n|| + ||b_n - a_n|| \leq 1/4n + 1/4n \leq 1/2n < 1/n$$

By the Archimedean Property, this implies that $a_n \to b$, and thus b is a limit point of A, a contradiction.

4. Let $A \subseteq \mathbb{R}$ be an open set and $f : A \to \mathbb{R}$. Show that the two following statements are equivalent:

(a) f is continuous

(b) for all $c \in \mathbb{R}$ the sets $E[f < c] = \{x \in A | f(x) < c\}$ and $E[f > c] = \{x \in A | f(x) > c\}$ are open

Solution:

(a $\Rightarrow$ b)

Note that $E[f < c] = f^{-1}((-\infty, c))$ and $E[f > c] = f^{-1}((c, \infty))$. Since f is continuous and $(-\infty, c)$ and (c, ∞) are open, this proves the implication

(b $\Rightarrow$ a)

We know that f is continuous iff the inverse image of any open set is continuous. Let $B \subseteq \mathbb{R}$ be an open set.

For any $y \in B$, there exists ε_y such that $B(y, \varepsilon_y) \subseteq B$. Thus, $\cup_{y \in B} B(y, \varepsilon_y) \subseteq B$. But if $z \in B, z \in B(z, \varepsilon_z)$, and thus $B \subseteq \cup_{y \in B} B(y, \varepsilon_y)$, which implies that $B = \cup_{y \in B} B(y, \varepsilon_y)$.

We also know that

$f^{-1}(\cup_{y \in B} B(y, \varepsilon_y)) = \cup_{y \in B} f^{-1}(B(y, \varepsilon_y))$

Since we are dealing with subsets of the real line, $B(y, \varepsilon_y) = (y - \varepsilon_y, y + \varepsilon_y)$. But notice that

$$\begin{aligned} (y - \varepsilon_y, y + \varepsilon_y) &= (-\infty, y + \varepsilon_y) \cap (y - \varepsilon_y, \infty) \\ \Rightarrow f^{-1}((y - \varepsilon_y, y + \varepsilon_y)) &= f^{-1}((-\infty, y + \varepsilon_y)) \cap f^{-1}((y - \varepsilon_y, \infty)) \\ &= E[f < y + \varepsilon_y] \cap E[f > y - \varepsilon_y] \end{aligned}$$

Since the finite intersection of open sets is open, this proves the result.

5. Let (X, d) be a metric space and $A \subseteq X$. Show that

$$\overline{A} = \{x \in X \mid d(x, A) = 0\}$$

where the distance between a point y and a set B is given by $d(y, B) = \inf_{b \in B}\{d(y, b)\}$.

Conclude that a set A is closed iff there exists a continuous function $f : X \to \mathbb{R}$ such that $A = f^{-1}(\{0\})$.

Solution:

Part 1: ($\overline{A} \subseteq \{x \in X \mid d(x, A) = 0\}$)

Let $y \in \overline{A}$ be arbitrary. If $y \in A$, we have that $0 \leq d(y, A) \leq d(y, y) = 0$, and thus $d(y, A) = 0$.

Suppose now that $y \in \overline{A} \setminus A$. We will first prove that there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $\forall n \in \mathbb{N}$ $x_n \in A$ and $x_n \to y$.

Suppose this is not the case. That means there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \bigcap A = \varnothing$. Particularly, $B(y, \varepsilon)^C$ is a closed set that contains A, and thus $\overline{A} \subseteq B(y, \varepsilon)^C$, which contradicts the fact that $y \in \overline{A}$, which finishes the proof for our auxiliary statement.

Take now $\{x_n\}_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$ $x_n \in A$ and $x_n \to y$. By the definition of convergence of sequences, we know that

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N}\ d(y, x_{n_0}) < \varepsilon$$

Particularly, $d(y, A) < d(y, x_{n_0}) < \varepsilon$. This implies that $d(y, A) = 0$.

Part 2: $(\{x \in X \mid d(x, A) = 0\}) \subseteq \overline{A}$

Suppose now that y is such that $d(y, A) = 0$. By the definition of infimum, this implies that.

$$\forall n \in \mathbb{N} \exists x_n \in A\ d(x_n, y) < 1/n$$

Let B be any closed set such that $A \subseteq B$. Since B is closed and $x_n \in A \subseteq B$, this implies that $y \in B$. Since B was taken arbitrary, this implies that y belongs to any closed set that contains A, and thus $y \in \overline{A}$, which finishes our proof.

We first need to show that $d(., A)$ is a continuous function. Notice that, for any $x, y \in X, z \in A$, we have that

$$\begin{aligned} d(x,z) \leq d(x,y) + d(y,z) &\Rightarrow d(x,A) \leq d(x,y) + d(y,z) \\ &\Rightarrow d(x,A) - d(x,y) \leq d(y,z) \\ &\Rightarrow d(x,A) - d(x,y) \leq d(y,A) \\ &\Rightarrow d(x,A) - d(y,A) \leq d(x,y) \end{aligned}$$

Similarly, we can prove that $d(y, A) - d(x, A) \leq d(x, y)$, and thus we have shown that $d(.A)$ is Lipschitz, and thus continuous. Furthermore, we have shown that $\overline{A} = (d(., A))^{-1}(0)$, which concludes this problem.

6. For some metric space (X, d), take any two sets $A, B \subset X$ such that $\text{int}A = \text{int}B = \emptyset$, and A is closed. Prove that $\text{int}(A \cup B) = \emptyset$.

Solution:

Towards a contradiction, assume $x \in \text{int}(A \cup B)$. By definition, this implies that there is some open ball $B_\epsilon(x) \subset A \cup B$. Consider the set $E = B_\epsilon(x) \backslash A = B_\epsilon(x) \cap A^c$. Since A is closed,

A^c is open. Since E is the finite intersection of two open sets, then it is open. We have two cases:

- $E = \emptyset$. This implies that $B_\epsilon(x) \subset A$, which implies that $x \in \text{int}A$, a contradiction.
- $E \neq \emptyset$. Then, for any $y \in E$, $y \in B$, so $E \subset B$. Since E is open, this implies that B has non-empty interior, a contradiction.

Since both cases lead to a contradiction, we conclude that $\text{int}(A \cup B) = \emptyset$.