Econ 204 – Problem Set 2[1]

Due Tuesday August 1 11:59PM, 2023

1. Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

2. Given $A, B \subseteq \mathbb{R}^n$, we define the sum of these two sets by:

$$A + B = \{a + b \mid a \in A, b \in B\}$$

   Prove or find a counterexample to the following statements:

   (a) If either A or B is an open set, then $A + B$ is an open set.

   (b) If both A and B are closed sets, $A + B$ is a closed set.

3. Show that

   (a) If $A \subseteq \mathbb{R}$ is open and $a_1, \ldots, a_n \in A$, then $A \setminus \{a_1, \ldots, a_n\}$ is open.

   (b) If A is open and $A \cap \overline{B} \neq \varnothing$, then $A \cap B \neq \varnothing$

   (c) If $\mathcal{A}$ is a collection of open subsets of $\mathbb{R}^n$, pairwise disjoint, that is, $A_\lambda \cap A_{\lambda'} = \varnothing$ if $\lambda \neq \lambda'$, then $\mathcal{A}$ is at most countable

   (d) The set of limit points of any set $A \subseteq \mathbb{R}^n$ is closed.

4. Let $A \subseteq \mathbb{R}$ be an open set and $f : A \to \mathbb{R}$. Show that the two following statements are equivalent:

   (a) f is continuous

   (b) for all $c \in \mathbb{R}$ the sets $E[f < c] = \{x \in A | f(x) < c\}$ and $E[f > c] = \{x \in A | f(x) > c\}$ are open

5. Let (X, d) be a metric space and $A \subseteq X$. Show that

$$\overline{A} = \{x \in X \mid d(x, A) = 0\}$$

   where the distance between a point y and a set B is given by $d(y, B) = \inf_{b \in B}\{d(y, b)\}$.

   Conclude that a set A is closed iff there exists a continuous function $f : X \to \mathbb{R}$ such that $A = f^{-1}(\{0\})$.

6. For some metric space (X, d), take any two sets $A, B \subset X$ such that $\text{int}A = \text{int}B = \emptyset$, and A is closed. Prove that $\text{int}(A \cup B) = \emptyset$.

[1]In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu