Econ 204 – Problem Set 3[1]

Solutions

1. Show that every open covering of $\mathbb{R}^n$ has a countable subcovering. *Hint: The countable union of finite sets is countable*

   Solution:

   Let $\mathcal{A}$ be an open covering of $\mathbb{R}^n$. Define, for each $k \in \mathbb{N}$, the following n-dimensional hypercube

$$C^k = [-k, k]^n$$

   Notice that since $C^k \subseteq R^n$, we have that $\mathcal{A}$ is an open covering of C^k. Thus there exist $A_1^k, \ldots, A_{m^k}^k$ a subcover of $\mathcal{A}$ such that

$$C^k \subseteq A_1^k \cup \cdots \cup A_{m^k}^k$$

   Let $\mathcal{A}^k = \{A_1^k, \ldots, A_{m^k}^k\}$. Note that $\mathcal{A}^k$ is a finite set.

   We can then take $\mathcal{A}^\infty = \cup_{k=1}^\infty \mathcal{A}^k$. Notice that $\mathcal{A}^\infty$ is the countable union of finite sets, and thus it is countable. It is also, by construction, a subcover of $\mathcal{A}$.

   Let $x \in \mathbb{R}^n$ be arbitrary. Since $||x||_\infty < \infty$, there exists $K \in \mathbb{N}$ such that $||x||_\infty < K$. This implies that $x \in C^K$. This proves that $\mathbb{R}^n = \cup_{k=1}^\infty C^k$. Particularly, $\mathcal{A}^\infty$ is a covering of $\mathbb{R}^n$. Since we have shown that it is countable, this finishes the proof.

2. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be bounded. Given $M > 0$, define $f_M : X \to \mathbb{R}$ by :

$$f_M(x) = \inf_{y \in X} \{f(y) + Md(x, y)\}$$

   Show that:

   (a) $\forall x \in X \; f_M(x) \leq f(x)$

   (b) Show that f_M is M-Lipschitz

   (c) Show that if f is Lipschitz and the lipschitz constant of f , M_f, is less or equal than M, then $f_M = f$

   (d) Show that, given $x \in X$ and $M < M'$, we have that $f_M(x) \leq f_{M'}(x)$.

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

(e) Show that when $M \to \infty$, then $f_M(x) \to f(x)$ in every point $x \in X$ such that f is continuous.

(f) Show that if f is continuous and X is compact,

$$\lim_{M\to\infty} \sup_{x\in X}\{d(f_M(x), f(x))\} = 0$$

Solution:

(a) Note that

$$\begin{aligned} f_M(x) &= \inf_{y\in X}\{f(y) + Md(x,y)\} \\ &\le f(x) + Md(x,x) \\ &= f(x) \end{aligned}$$

(b) Note that, for any given $x, z \in X$

$$\begin{aligned} f(z) + Md(x,z) &\le f(z) + Md(y,z) + Md(x,y) \quad \text{(triangle inequality)} \\ \Rightarrow \inf_z\{f(z) + Md(x,z)\} &\le \inf_z\{f(z) + Md(y,z)\} + Md(x,y) \quad \text{(taking inf over z)} \\ \Rightarrow f_M(x) &\le f_M(y) + Md(x,y) \\ \Rightarrow f_M(x) - f_M(y) &\le Md(x,y) \end{aligned}$$

Similarly, $f_M(y) - f_M(x) \le Md(x,y)$, and thus

$$|f_M(x) - f_M(y)| \le Md(x,y)$$

Which proves the result.

(c) We already know that $\forall x \in X$ $f_M(x) \le f(x)$. Note now that for any given $x, z \in X$,

$$\begin{aligned} f(x) - f(z) &\leq M_f d(x,z) \\ &\leq M d(x,z) \\ \Rightarrow \forall z \in X \ f(x) &\leq f(z) + M d(x,z) \\ \Rightarrow \ f(x) &\leq \inf_{y \in X} \{f(y) + M d(x,y)\} \\ \Rightarrow \ f(x) &\leq f_M(x) \end{aligned}$$

and thus $f(x) = f_M(x) \forall x \in X$, which concludes our proof.

(d) Fix $x \in X$. Note that

$$\begin{aligned} \forall z \in X \ f(z) + M d(x,z) &\leq f(z) + M' d(x,z) \\ \Rightarrow \inf_{y \in X} \{f(y) + M d(x,y)\} &\leq \inf_{y \in X} \{f(y) + M' d(x,y)\} \\ \Rightarrow f_M(x) &\leq f_{M'}(x) \end{aligned}$$

(e) Let $x \in X$ be a point in which f is continuous. Since f is bounded, we know that there exists $C > 0$ such that $|f(x)| \leq C$. Since f is continuous at x, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} d(x,y) < \delta &\Rightarrow |f(x) - f(y)| < \varepsilon \\ &\Rightarrow f(x) - \varepsilon < f(y) \\ &\Rightarrow f(x) - \varepsilon < f(y) + M d(x,y) \end{aligned}$$

If $y \notin B(x,\delta)$, note that $f(y) + M d(x,y) \geq f(y) + M\delta \geq M\delta - C$. For $M > 2C/\delta$, this implies that $f(y) + M d(x,y) > f(x) > f(x) - \varepsilon$

This implies that if $M > 2C/\delta$, then $f_M(x) \geq f(x) - \varepsilon$. Since $\varepsilon > 0$ was taken arbitrary, this implies that $f_M(x) \geq f(x)$. Combining this with a), this implies that

$$\forall M > 2C/\delta \ f(x) \geq f_M(x) \geq f(x) - \epsilon$$

Since $\varepsilon > 0$ was taken arbitrary, we get that $lim_{M \to \infty} f_M(x) \to f(x)$.

(f) If f is continuous and X is compact, f is uniformly continuous. In this case we can repeat the argument above, but noticing that the bounds using continuity now hold for all x simultaneously.

3. Let $U \subseteq \mathbb{R}^d$ be an open set and $f : [0,1] \to U$ be continuous. For each $n \in \mathbb{N}$, define the n-polygonal approximation of f to be the function $\gamma_n : [0,1] \to \mathbb{R}^d$ given by:

$$\gamma_n(t) = f\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right)$$

where $i \in \{1, \ldots, n\}$ is such that $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$.

(a) Show that γ_n is continuous for all $n \in \mathbb{N}$.

Solution:

Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $a \in [0,1]$ and $j \in \mathbb{N}$ be such that $a \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$. Note that

$$\begin{aligned}
\gamma_n(a) - f\left(\frac{j-1}{n}\right) &= f\left(\frac{j-1}{n}\right) + n\left(a - \frac{j-1}{n}\right)\left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right) - f\left(\frac{j-1}{n}\right) \\
&= n\left(a - \frac{j-1}{n}\right)\left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right)
\end{aligned}$$

A similar inequality holds for the distance between the images of a and j/n. This will be useful to find out the δ in the definition of continuity. I encourage you to think about what we have just done graphically. Let now $M = \max_{x\in[0,1]} |f(x)|$ and define

$$\delta = \min\left(\frac{\varepsilon}{2Mn}, \frac{1}{n}\right)$$

and let $a, b \in [0,1]$ be such that $|a-b| < \delta$. Without loss of generality, assume that $a < b$. We then have two cases:

Case 1: $\exists j \in \{1, \ldots, n\}$ such that $a, b \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$.

We then have that :

$$\begin{aligned}
|\gamma_n(b) - \gamma_n(a)| &= \left| f\left(\frac{j-1}{n}\right) + n\left(b - \frac{j-1}{n}\right)\left(f\left(\frac{j}{n}\right)\right) - f\left(\frac{j-1}{n}\right) - n\left(a - \frac{j-1}{n}\right)\left(f\left(\frac{j}{n}\right)\right)\right| \\
&= \left| n\,(b-a)\, f\left(\frac{j}{n}\right)\right| \\
&< n\frac{\varepsilon}{2Mn}\left| f\left(\frac{j}{n}\right)\right| \\
&< \varepsilon
\end{aligned}$$

Case 2: There exists $j \in \{1, \ldots, n\}$ such that $\max\left(|b - \frac{j}{n}|, |a - \frac{j}{n}|\right) < \delta$. By the inequality we proved above, we have that

$$
\begin{aligned}
\left|\gamma_n(b) - \gamma_n(a)\right| &= \left| f(b) - f\left(\frac{j}{n}\right) + f\left(\frac{j}{n}\right) - f(a) \right| \\
&\le \left| n\left(a - \frac{j}{n}\right)\left(f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right)\right| + \left| n\left(b - \frac{j}{n}\right)\left(f\left(\frac{j+1}{n}\right) - f\left(\frac{j-1}{n}\right.\right.\right. \\
&< n\delta \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right| + n\delta \left| f\left(\frac{j+1}{n}\right) - f\left(\frac{j-1}{n}\right)\right| \\
&< n\frac{\varepsilon}{2Mn}\left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right| + n\frac{\varepsilon}{2Mn}\left| f\left(\frac{j+1}{n}\right) - f\left(\frac{j-1}{n}\right)\right| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

where we have used the fact that, for any $j \in \{1, \ldots, n\}$,

$$
\begin{aligned}
\left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right| &\le \left| f\left(\frac{j}{n}\right)\right| + \left| f\left(\frac{j-1}{n}\right)\right| \\
&\le 2M
\end{aligned}
$$

(b) Show that there exists $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0 \gamma_n(t) \in U$ for all $t \in [0, 1]$.

Solution:

First note that, for any $t \in [0, 1]$ and $n \in \mathbb{N}$, we have that

$$
\begin{aligned}
|f(t) - \gamma_n(t)| &= \left| f(t) - f\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right)\right| \\
&\le \left| f(t) - f\left(\frac{i-1}{n}\right)\right| + n\left(t - \frac{i-1}{n}\right)\left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| \\
&\le \left| f(t) - f\left(\frac{i-1}{n}\right)\right| + n\left(\frac{i}{n} - \frac{i-1}{n}\right)\left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| \\
&= \left| f(t) - f\left(\frac{i-1}{n}\right)\right| + \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|
\end{aligned}
$$

Since U is open, this means that $\forall t \in [0,1]$ $\exists \varepsilon_t > 0$ such that $B(f(t), \varepsilon_t) \subseteq U$. Since f is continuous and $[0,1]$ is compact, we have that $f([0,1])$ is a compact set. Notice that the collection $B(f(t), \varepsilon_t/3)$ is an open covering of $f([0,1])$. By definition (or a characterization) of compactness, this means that there exists $t_1, \ldots, t_N$ such that

$$\begin{aligned} f([0,1]) &\subseteq B(f(t_1), \varepsilon_{t_1}/3) \cup \cdots \cup B(f(t_N), \varepsilon_{t_N}/3) \\ &\subseteq U \end{aligned}$$

Let now $\varepsilon = \min(\varepsilon_{t_1}/3, \ldots, \varepsilon_{t_N}/3)/2$. Since f is a continuous function and $[0,1]$ is a compact set, f is uniformly continuous, thus there exists $\delta > 0$ such that

$$|a - b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon$$

Let n_0 be such that $1/n_0 < \delta$, and let $t \in [0,1], n > n_0$ be arbitrary. Let k be such that $f(t) \in B(f(t_k), \varepsilon_{t_k}/3)$. We then have that

$$\begin{aligned} |\gamma_n(t) - f(t_k)| &\leq |\gamma_n(t) - f(t)| + |f(t) - f(t_k)| \\ &\leq \left| f(t) - f\left(\frac{i-1}{n}\right) \right| + \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right| + |f(t) - f(t_k)| \\ &< \varepsilon + \varepsilon + \varepsilon_{t_k}/3 \\ &< 3 * \varepsilon_{t_k}/3 \\ &= \varepsilon_{t_k} \end{aligned}$$

Thus $\gamma(t) \in B(f(t_k), \varepsilon_{t_k}) \subseteq U$, which finishes the proof.

4. Let (X, d) be a metric space. Given $x \in X$, we define the connected component of x in X as the set

$$C(x) = \bigcup_{\substack{U \subseteq X \text{ s.t } x \in U \\ U \text{ is connected}}} U$$

Prove that:

(a) For every $x \in X$, $C(x)$ is a non-empty connected set.

Solution:

Note that $\{x\}$ is a connected set, thus the union is taken over a non-empty collection, thus $x \in C(x)$ and $C(x)$ is non-empty.

Suppose now, by contradiction, that $C(x)$ is not connected. We then have that there exists A, B open, disjoint, non-empty sets such that $C(x) \cap A \neq \varnothing$, $C(x) \cap B \neq \varnothing$ and $C(x) \subseteq A \cup B$.

Since $x \in C(x)$, we have that $x \in A$ or $x \in B$, but not both, since they are disjoint. Without loss of generality, let $x \in A$.

Let $y \in C(x) \cap B$, which exists because $C(x) \cap B \neq \varnothing$. Since $y \in C(x)$, we have that there exists U_y connected such that $x \in U_y, y \in U_y, U_y \subseteq X$ and U_y is connected.

Let now

$$E_1 = \{z \in U_y \mid z \in A\}$$
$$E_2 = \{z \in U_y \mid z \in B\}$$

We know that $x \in E_1$ and $y \in E_2$. We also have that $U_y \subseteq C(x) \subseteq A \cup B$, so that $E_1 \cup E_2 = U_y$. This implies that U_y is not a connected set, which is a contradiction. Thus $C(x)$ is a connected set.

(b) For every two elements $x, y \in X$, they either share a connected component $C(x) = C(y)$ or their connected components are disjoint $C(x) \cap C(y) = \varnothing$.

Solution:

If $C(x) \cap C(y) = \varnothing$, there is nothing to be done. If this is not the case, then there exists $z \in C(x) \cap C(y)$.

We will prove now that $C(x) \cup C(y)$ is connected. Assume, by contradiction, that this is not the case. Then there exists A, B open, non-empty disjoint sets such that $(C(x) \cup C(y)) \cap A \neq \varnothing$, $(C(x) \cup C(y)) \cap B \neq \varnothing$ and $C(x) \cup C(y) \subseteq A \cup B$.

Since $z \in C(x) \cup C(y)$, we have that $z \in A \cup B$, without loss of generality assume that $z \in A$. Since $(C(x) \cup C(y)) \cap B \neq \varnothing$, there exists $w \in (C(x) \cup C(y)) \cap B$, without loss of generality assume that $w \in C(x) \cap B$. Note also that $z \in C(x) \cap A$. This contradicts the fact that $C(x)$ is connected, so it must be the case that $C(x) \cup C(y)$ is connected.

By definition, this means that $C(y) \subseteq C(x) \cup C(y) \subseteq C(x)$ and $C(x) \subseteq C(x) \cup C(y) \subseteq C(y)$, and thus $C(x) = C(y)$.

(c) Conclude that there exists a subset $\mathcal{A} \subseteq X$ such that $X = \dot{\bigcup}_{x \in \mathcal{A}} C(x)$, where $\dot{\bigcup}$ represents the disjoint union.

Solution:

Define the following equivalence relation on X: $x \sim y \iff C(x) = C(y)$, and consider the partition $X/\sim$. Let $\mathcal{A}$ be a set formed by taking one element of each set of $X/\sim$. By construction, $X = \dot{\cup}_{x\in\mathcal{A}} C(x)$, and the union is disjoint by b).

5. Let X be a compact set and $\Gamma : X \to 2^X$ be a non-empty, compact-valued upper-hemicontinuous correspondence. Show that if $C \subseteq X$ is compact, then $\Gamma(C)$ is compact.

Solution:

Let $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ be an open cover of $\Gamma(C)$. Let $x \in C$. Notice that $\mathcal{A}$ is also an open cover of $\Gamma(x)$, since $\Gamma(x) \subseteq \Gamma(C)$, and $\Gamma(x)$ is compact since Γ is compact-valued. So there exists $A_{\lambda_{x1}}, \ldots, A_{\lambda_{xn}}$ such that $\Gamma(x) \subseteq A_{\lambda_{x1}} \cup \cdots \cup A_{\lambda_{xn}}$. Set $A^x = A_{\lambda_{x1}} \cup \cdots \cup A_{\lambda_{xn}}$. Since $A_{\lambda_{xi}}$ is open for each i, A^x is an open set. Then since Γ is uhc, there is an open set $U_x \ni x$ such that for all $z \in U_x, \Gamma(z) \subseteq A^x$.

By construction, $\{U_x : x \in C\}$ is an open cover of C. Since C is compact, there exist $x_1, \ldots, x_m \in C$ such that $C \subseteq U_{x_1} \cup \cdots \cup U_{x_m}$. Thus $\Gamma(C) \subseteq \cup_{i=1}^m A^{x_i}$, which proves that it is compact.

6. Let (X, d) be a compact metric space.

   (a) Show that there exists A an at most countable subset of X such that $\overline{A} = X$.

   (b) We say that $x \in X$ is an isolated point if there exists $\delta > 0$ such that $B(x, \delta) = \{x\}$. Show that the set of isolated points of X is empty, finite or countable.

Solution:

(a) Note that for any $n \in \mathbb{N}$, we have that $\{B(x, 1/n)\}_{x\in X}$ is an open covering of X. Since X is compact, this means that there exists $\{x_1^n, \ldots, x_{N_n}^n\}$ such that

$$X \subseteq \bigcup_{k=1,\ldots,N_n} B(x_k^n, 1/n)$$

Let $A = \bigcup_{n\in\mathbb{N}} \{x_1^n, \ldots, x_{N_n}^n\}$. Since A is a countable union of finite sets, A is at most countable. We will now prove that $\overline{A} = X$.

Let $y \in X$ be arbitrary. Since $\forall n \in \mathbb{N}$ $y \in X \subseteq \bigcup_{k=1,\ldots,N_n} B(x_k^n, 1/n)$, we have that $y \in B(x_k^n, 1/n)$ for some $i \in \{1, \ldots, N_n\}$, denote such x_i^n by y_n.

Notice that $\forall n \in \mathbb{N}$ $y_n \in A$. Furthermore, $d(x, y_n) < 1/n$, and thus $y_n \to x$. We thus have showed that every element of X is the limit of a sequence in A, and thus $\overline{A} = X$.

(b) Let K be the set of isolated points of X. For each $x \in X$, there exists $\delta_x > 0$ such that $B(x, \delta_x) = \{x\}$. Let A be a countable set such as in a). Since $\overline{A} = X$, this means that there $B(x, \delta_x) \cap A \neq \varnothing$, and thus $x \in A$. We have just proved that $K \subseteq A$, and thus K is at most countable, which finishes our proof.