Econ 204 – Problem Set 3[1]

Due Friday August 4 11:59PM, 2023

1. Show that every open covering of $\mathbb{R}^n$ has a countable subcovering. *Hint: The countable union of finite sets is countable*

2. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be bounded. Given $M > 0$, define $f_M : X \to \mathbb{R}$ by :

$$f_M(x) = \inf_{y \in X}\{f(y) + Md(x, y)\}$$

Show that:

   (a) $\forall x \in X\ f_M(x) \leq f(x)$
   (b) Show that f_M is M-Lipschitz
   (c) Show that if f is Lipschitz and the lipschitz constant of f , M_f, is less or equal than M, then $f_M = f$
   (d) Show that, given $x \in X$ and $M < M'$, we have that $f_M(x) \leq f_{M'}(x)$.
   (e) Show that when $M \to \infty$, then $f_M(x) \to f(x)$ in every point $x \in X$ such that f is continuous.
   (f) Show that if f is continuous and X is compact,

$$\lim_{M\to\infty} \sup_{x\in X}\{d(f_M(x), f(x))\} = 0$$

3. Let $U \subseteq \mathbb{R}^d$ be an open set and $f : [0, 1] \to U$ be continuous. For each $n \in \mathbb{N}$, define the n-polygonal approximation of f to be the function $\gamma_n : [0, 1] \to \mathbb{R}^d$ given by:

$$\gamma_n(t) = f\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right)$$

where $i \in \{1, \ldots, n\}$ is such that $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$.

   (a) Show that γ_n is continuous for all $n \in \mathbb{N}$.
   (b) Show that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 \gamma_n(t) \in U$ for all $t \in [0, 1]$.

4. Let (X, d) be a metric space. Given $x \in X$, we define the connected component of x in X as the set

[1] In case of any problems with the solution to the exercises please email brunosmaniotto@berkeley.edu

$$C(x) = \bigcup_{\substack{U \subseteq X \text{ s.t } x \in U \\ U \text{ is connected}}} U$$

Prove that:

(a) For every $x \in X$, $C(x)$ is a non-empty connected set.

(b) For every two elements $x, y \in X$, they either share a connected component $C(x) = C(y)$ or their connected components are disjoint $C(x) \cap C(y) = \varnothing$.

(c) Conclude that there exists a subset $\mathcal{A} \subseteq X$ such that $X = \dot{\bigcup}_{x \in \mathcal{A}} C(x)$, where $\dot{\cup}$ represents the disjoint union.

5. Let X be a compact set and $\Gamma : X \to 2^X$ be a non-empty, compact-valued upper-hemicontinuous correspondence. Show that if $C \subseteq X$ is compact, then $\Gamma(C)$ is compact.

6. Let (X, d) be a compact metric space.

   (a) Show that there exists A an at most countable subset of X such that $\overline{A} = X$.

   (b) We say that $x \in X$ is an isolated point if there exists $\delta > 0$ such that $B(x, \delta) = \{x\}$. Show that the set of isolated points of X is empty, finite or countable.