Econ 204 – Problem Set 4 Suggested Solutions

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Matrix Representation of Linear Transformations

Problem 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (4x - 2y, x + y)$. Let V be the standard basis and $W = \{(5, 3), (1, 1)\}$ be another basis of $\mathbb{R}^2$.

1. Find $Mtx_V(T)$.
2. Find $Mtx_W(T)$.
3. Compute $T(4, 3)$ using the matrix representation of W.

Solution

1. $$Mtx_V(T) = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$
2. First,
$$Mtx_{V,W}(id) = \begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}$$
and $Mtx_{W,V}(id) = Mtx_{V,W}(id)^{-1}$, then
$$Mtx_{W,V}(id) = \begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -3 & 5 \end{pmatrix}$$
Finally
$$Mtx_W(T) = Mtx_{W,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,W}(id) = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}$$
3. We know that $Mtx_W(T) \cdot crd_W(4, 3) = crd_W(T(4, 3))$. Solving $(4, 3) = \alpha(5, 3) + \beta(1, 1)$ yields $\alpha = 1/2$ and $\beta = 3/2$. Then
$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1\backslash 2 \\ 3\backslash 2 \end{pmatrix} = \begin{pmatrix} 3\backslash 2 \\ 5\backslash 2 \end{pmatrix}$$
And $\frac{3}{2} \cdot (5, 3) + \frac{5}{2}(1, 1) = (10, 7) = T(4, 3)$.

Invertibility

Problem 2

Let V be a finite dimensional vector space with dimension $n > 1$. Let $L(V, V)$ be the set of all linear transformation from V to V, which is a vector space (you don't have to prove this). Consider $C \subset L(V, V)$, the set of all non-invertible linear transformations from V to V. Is C a subspace of $L(V, V)$? Prove or provide a counterexample.

Solution

No. Let $v = (v_1, ..., v_n) \in V$, with v_n non-zero for all n. Define T as $Tv = (v_1, v_2, ..., v_{n-1}, 0)$ and S as $Sv = (0, 0, ..., 0, v_n)$. Note that T and S are not invertible, since $(0, 0, ..., 0, v_n) \in \ker T$, and $(v_1, v_2, ..., v_{n-1}, 0) \in \ker S$. However, note that $(T + S)v = v$, that is, $T + S$ is the identity mapping, which is invertible. Then, C is not closed under addition and, therefore, is not a vector subspace of $L(V, V)$.

Problem 3

The norm on the space of square matrices $R^{n\times n}$ is defined as follows: for every $A \in R^{n\times n}$

$$||A|| = \sup\{\ ||Ax||_{R^n} : x \in R^n \text{ and } ||x||_{R^n} = 1\}$$

We can also define a metric d on the space of $n \times n$ matrices using this norm:

$$d(A, B) = ||A - B||$$

Take as given that $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous. Use the continuity of the determinant to prove that the set of all invertible matrices is an open, dense subset of all square matrices.
Note: if you are not familiar with the norm/metric on the space of square matrices, in this problem you only need to use the following properties

- for any constant $c \in R$, $||cA|| = |c|||A||$
- the norm of the identity matrix I is 1, $||I|| = 1$

Also, to show that the set $S \subset X$ is dense in X, for any $x \in X$, construct a sequence $\{s_n\}_{n\in N}, s_n \in S$ for all n, such that $s_n \to x$ in the respective metric.

Solution

Remember that a matrix is invertible iff its determinant is nonzero. So the set of all invertible matrices is exactly $\det^{-1}(\mathbb{R} \setminus \{0\})$. Observe that $\mathbb{R} \setminus \{0\}$ is open. By continuity of the determinant, the pre-image of an open set is open, so the set of invertible matrices is open. Also note that this says the set of $n \times n$ singular matrices is a closed subset of $\mathbb{R}^{n\times n}$.

Next, fix some singular matrix A. Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(t) = \det(A - tI)$$

Then note that $g(t) = 0$ if and only if t is a real eigenvalue of A. Since A is singular, 0 is an eigenvalue of A, so $g(0) = 0$. Since A has at most only finitely many eigenvalues (real or complex), there can be at most finitely many points $t_1, \dots, t_k \in \mathbb{R}$ such that $g(t_i) = 0$. Thus there exists some $\varepsilon > 0$ such that $g(t) \neq 0$ for all $t \in (0, \varepsilon)$. Take the sequence $t_n = \frac{1}{n}$ and choose N such that $n > N$ implies $t_n < \varepsilon$, so $g(t_n) \neq 0$. This says that the matrix $A - t_n I$ is invertible. Further,

$$\begin{aligned} d(A, A - t_n I) &= \|A - (A - t_n I)\| \\ &= \|t_n I\| \\ &= |t_n| \to 0 \end{aligned}$$

Hence $A - t_n I$ is a sequence of invertible matrices that converges to A. Thus A is a limit point of the set of all invertible matrices.

Invariant Subspaces

Problem 4

1. Let $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ be given by
$$T(x_1, x_2) = (-x_2, x_1)$$
Find the eigenvalues and eigenvectors of T. Explain the intuition.

2. Now suppose the field is $\mathbb{C}$ instead of $\mathbb{R}$, so consider $T \in L(\mathbb{C}^2, \mathbb{C}^2)$ given by
$$T(z_1, z_2) = (-z_2, z_1)$$
where here $z_1, z_2 \in \mathbb{C}$. Find the eigenvalues and eigenvectors of T. Note that an eigenvalue $\lambda \in \mathbb{C}$ and an eigenvector $z \in \mathbb{C}^2$

Solution

By definition, $\lambda \in F$ is an eigenvalue of T if there exists a vector $x \neq 0$ such that

$$T(x) = \lambda x$$

1. For $F = \mathbb{R}$, $\lambda \in \mathbb{R}$ is an eigenvalue for T if
$$T(x_1, x_2) = (-x_2, x_1) = \lambda(x_1, x_2)$$
for some $(x_1, x_2) \neq (0, 0)$. Thus
$$-x_2 = \lambda x_1 \text{ and } x_1 = \lambda x_2$$
Note that if (x_1, x_2) satisfies these equations, then $x_1 = 0 \iff x_2 = 0$. Since $(x_1, x_2) \neq (0, 0)$, we must have $x_1 \neq 0$ and $x_2 \neq 0$. This implies
$$-x_2 = \lambda x_1 = \lambda^2 x_2$$
Since $x_2 \neq 0$, this implies $\lambda^2 = -1$. There is no $\lambda \in \mathbb{R}$ such that $\lambda^2 = -1$, so T has no eigenvalues or eigenvectors.

Notice that the linear transformation T is counterclockwise rotation of vectors around the origin in $\mathbb{R}^2$ by 90^o. This linear transformation has no eigenvectors or eigenvalues because no vector is mapped to a scaled version of itself by this transformation (where the scalar is a real number).

2. For $F = \mathbb{C}$, $\lambda \in \mathbb{C}$ is an eigenvalue for T if

$$T(z_1, z_2) = (-z_2, z_1) = \lambda(z_1, z_2)$$

for some $(z_1, z_2) \neq (0, 0)$. Thus

$$-z_2 = \lambda z_1 \text{ and } z_1 = \lambda z_2$$

As in (a), note that $z_1 = 0 \iff z_2 = 0$, so since $(z_1, z_2) \neq (0, 0)$ we must have $z_1 \neq 0$ and $z_2 \neq 0$. Then as above this requires

$$-z_2 = \lambda z_1 = \lambda^2 z_2$$

Since $z_2 \neq 0$ this implies $\lambda^2 = -1$. Thus $\lambda = i$ and $\lambda = -i$ are eigenvalues of T.
To find the eigenvectors, first consider $\lambda = i$. Then

$$-z_2 = iz_1 \text{ and } z_1 = iz_2$$

So the corresponding eigenvectors are of the form $(c, -ci)$ for all $c \in \mathbb{C} \setminus \{0\}$.
Then consider $\lambda = -i$. In this case,

$$-z_2 = -iz_1 \text{ and } z_1 = -iz_2$$

So the corresponding eigenvectors are (c, ci) for all $c \in \mathbb{C} \setminus \{0\}$.

Problem 5

Let A be an $n \times n$ matrix.

1. Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$.

2. Show that if λ is an eigenvalue of the matrix A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}.

3. Find an expression for $\det(A)$ in terms of the eigenvalues of A.

4. The *eigenspace* of an eigenvalue λ_i of A is the kernel of $A - \lambda_i I$ (all $x \in R^n$ such that $(A - \lambda_i I)x = 0$). Show that the eigenspace of any eigenvalue λ_i of A is a vector subspace of R^n.

Solution

1. We use induction to show not only that λ^k is an eigenvalue of A^k, but also that any eigenvector v corresponding to the eigenvalue λ for A also corresponds to λ^k for A^k. The base step $(k = 1)$ is trivial. For the induction step, assume $Av = \lambda v$ and $A^k v = \lambda^k v$. Now consider $A^{k+1}v$:

$$A^{k+1}v = A^k(Av) = A^k(\lambda v) = \lambda(A^k v) = \lambda(\lambda^k v) = \lambda^{k+1}v$$

2. Let $Tv = \lambda v$. Premultiply both sides by T^{-1}:

$$T^{-1}Tv = T^{-1}\lambda v \implies v = \lambda T^{-1}v \implies T^{-1}v = (1/\lambda)v$$

3. The characteristic polynomial of A is given by $c(\lambda) = \det(A - \lambda I)$. The eigenvalues are the roots of this function; this is,

$$c(\lambda) = \det(A - \lambda I) = (-1)^n \prod_i (\lambda - \lambda_i)$$

Hence setting $\lambda = 0$ we get

$$\det(A) = \prod_i \lambda_i$$

so the determinant of a matrix is the product of its eigenvalues.

4. It follows immediately from the more general result that was given in class, that for any linear transformation T, the kernel of T is a vector subspace. We use the fact that

$$T(x) = (A - \lambda_i I)x$$

is a linear transformation.
On the other hand, to verify directly that $W \subseteq X$ is a vector subspace of the vector space X, we need to show that for all $w_1, w_2 \in W$ and all $\alpha_1, \alpha_2 \in F$, $\alpha_1 w_1 + \alpha_2 w_2 \in W$.
So here to show that $\ker T$ is a vector subspace, let $x_1, x_2 \in \ker T$. Then for any α_1, α_2,

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) = \alpha_1 0 + \alpha_2 0 = 0$$

So $\alpha_1 x_1 + \alpha_2 x_2 \in \ker T$.

Linear Maps between Normed Spaces

Problem 6

Let X be a normed vector space. Let $T : X \to R$ be a linear map. Prove that T is bounded if and only if $T^{-1}(\{0\})$ is closed.

Solution

For any continuous function between any two topological spaces, the inverse image of any closed set is closed. If T is bounded then it is continuous. Since $\{0\}$ is a closed subset of $\mathbb{R}$, its inverse image under T is therefore closed.
Another version of the proof (contrapositive): suppose that $T^{-1}(\{0\})$ is not closed. Then there exist $x_n \in X$ and $y \in X$ such that $T(x_n) = 0, x_n \to y$, and $T(y) \neq 0$. Since T is linear, $y \neq 0$. Since $T(x_n) = 0, x_n - y \neq 0$.
For each n,

$$\|T\|_{B(X,R)} \geq \frac{|T(x_n - y)|}{\|x_n - y\|_X} = \frac{|T(y)|}{\|x_n - y\|_X}$$

The numerator is nonzero and independent of n. As $n \to \infty$, the denominator tends to zero. Therefore the ratio tends to infinity and T is not bounded.
For the converse, suppose T is not bounded (contrapositive again). Then for each n there exists $x_n \in X$ such that $\|x_n\| = 1$ and $T(x_n) > n$. Then fix $x \in X$ such that $T(x) \neq 0$, and without loss of generality take $T(x) = 1$. Let

$$y_n = x - \frac{1}{T(x_n)} x_n \quad \forall n$$

Note that $T(x_n) > n > 0$ for each n, so y_n is well-defined for each n. Then by construction, $T(y_n) = 0$ for each n, so $y_n \in T^{-1}(\{0\})$ for each n, and $y_n \to x$. But $T(x) \neq 0$, so $x \notin T^{-1}(\{0\})$. Thus $T^{-1}(\{0\})$ is not closed.