Econ 204 – Problem Set 4

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Matrix Representation of Linear Transformations

Problem 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (4x - 2y, x + y)$. Let V be the standard basis and $W = \{(5, 3), (1, 1)\}$ be another basis of $\mathbb{R}^2$.

1. Find $Mtx_V(T)$.
2. Find $Mtx_W(T)$.
3. Compute $T(4, 3)$ using the matrix representation of W.

Invertibility

Problem 2

Let V be a finite dimensional vector space with dimension $n > 1$. Let $L(V, V)$ be the set of all linear transformation from V to V, which is a vector space (you don't have to prove this). Consider $C \subset L(V, V)$, the set of all non-invertible linear transformations from V to V. Is C a subspace of $L(V, V)$? Prove or provide a counterexample.

Problem 3

The norm on the vector space of square matrices $R^{n\times n}$ is defined as follows: for every $A \in R^{n\times n}$

$$||A|| = \sup\{\ ||Ax||_{R^n} : x \in R^n \text{ and } ||x||_{R^n} = 1\}$$

Recall that this norm is an operator norm defined in lecture 10. We can also define a metric d on the vector space of $n \times n$ matrices using this norm:

$$d(A, B) = ||A - B||$$

Take as given that $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous in this metric. Use the continuity of the determinant to prove that the set of all invertible matrices is an open, dense subset of all square matrices. Hint: in this problem you will only need to use the following properties of the norm

- for any constant $c \in R$, $||cA|| = |c|||A||$
- the norm of the identity matrix I is 1, $||I|| = 1$

Invariant Subspaces

Problem 4

1. Let $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ be given by
$$T(x_1, x_2) = (-x_2, x_1)$$
Find the eigenvalues and eigenvectors of T. Explain the intuition.

2. Now suppose the field is $\mathbb{C}$ instead of $\mathbb{R}$, so consider $T \in L(\mathbb{C}^2, \mathbb{C}^2)$ given by
$$T(z_1, z_2) = (-z_2, z_1)$$
where here $z_1, z_2 \in \mathbb{C}$. Find the eigenvalues and eigenvectors of T. Note that an eigenvalue $\lambda \in \mathbb{C}$ and an eigenvector $z \in \mathbb{C}^2$

Problem 5

Let A be an $n \times n$ matrix.

1. Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$.

2. Show that if λ is an eigenvalue of the matrix A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}.

3. Find an expression for $\det(A)$ in terms of the eigenvalues of A.

4. The *eigenspace* of an eigenvalue λ_i of A is the kernel of $A - \lambda_i I$ (all $x \in R^n$ such that $(A - \lambda_i I)x = 0$). Show that the eigenspace of any eigenvalue λ_i of A is a vector subspace of R^n.

Linear Maps between Normed Spaces

Problem 6

Let X be a normed vector space. Let $T : X \to R$ be a linear map. Prove that T is bounded if and only if $T^{-1}(\{0\})$ is closed.