Econ 204 – Problem Set 5 Suggested Solutions

GSI - Anna Vakarova

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1 Taylor Theorem and Mean Value Theorem

Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Suppose $f'(0) = f'(1) = 2$ and $\forall x \in [0, 1]$, $|f''(x)| \leq 4$.

a. Prove that $|f(1) - f(0)| \leq 4$.

b. Prove that $|f(1) - f(0)| \leq 3$.

(Hint: use Taylor's Theorem)

Solution

a. By Taylor's Theorem,

$$f(1) = f(0) + f'(0) + \frac{f''(a)}{2}$$

for some $a \in (0, 1)$. This implies

$$f(1) - f(0) = 2 + \frac{f''(a)}{2}$$

since $f'(0) = 2$, so

$$|f(1) - f(0)| = |2 + \frac{f''(a)}{2}| \leq 2 + \frac{|f''(a)|}{2} \leq 4$$

b. First, consider Taylor's Theorem on the interval $[0, \frac{1}{2}]$. This implies

$$f(\frac{1}{2}) = f(0) + \frac{f'(0)}{2} + \frac{f''(b)}{8}$$

for some $b \in (0, \frac{1}{2})$. Thus, since $f'(0) = 2$,

$$f(\frac{1}{2}) - f(0) = 1 + \frac{f''(b)}{8}$$

This implies

$$|f(\frac{1}{2}) - f(0)| \leq |1 + \frac{f''(b)}{8}| \leq 1 + |\frac{f''(b)}{8}| \leq 1 + \frac{1}{2}$$

Similarly, considering Taylor's Theorem on the interval $[\frac{1}{2}, 1]$,

$$f(\frac{1}{2}) = f(1) - \frac{f'(1)}{2} + \frac{f''(c)}{8}$$

for some $c \in (\frac{1}{2}, 1)$. Using $f'(1) = 2$, this implies

$$f(\frac{1}{2}) - f(1) = -1 + \frac{f''(c)}{8}$$

and

$$|f(\frac{1}{2}) - f(1)| \leq 1 + \frac{|f''(c)|}{8} \leq 1 + \frac{1}{2}$$

Then

$$|f(1) - f(0)| \leq |f(1) - f(\frac{1}{2})| + |f(\frac{1}{2}) - f(0)| \leq 1 + \frac{1}{2} + 1 + \frac{1}{2} = 3$$

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 (twice continuously differentiable) function. The function and its second derivative are bounded, namely there exist $M, N > 0$ such that $\sup_{x\in\mathbb{R}} |f(x)| \leq M$ and $\sup_{x\in\mathbb{R}} |f''(x)| \leq N$. Show that $\sup_{x\in\mathbb{R}} |f'(x)| \leq 2\sqrt{MN}$.

Solution

Fix an arbitrary $x \in \mathbb{R}$. Then, using Taylor's theorem for every $y \in \mathbb{R}$ there exists ξ between x and y such that

$$\begin{aligned} f(y) &= f(x) + f'(x)(y-x) + \frac{1}{2} f''(\xi)(y-x)^2 \\ &\leq f(x) + f'(x)(y-x) + \frac{1}{2} N(y-x)^2. \end{aligned}$$

Since $f(x) - f(y) \leq 2M$, then $\frac{1}{2}N(y-x)^2 + f'(x)(y-x) + 2M \geq 0$ for every $y \in \mathbb{R}$. Therefore, the quadratic polynomial

$$g(t) = \frac{1}{2}Nt^2 + f'(x)t + 2M,$$

is nonnegative for all $t \in \mathbb{R}$. Consequently, its $\Delta = (f'(x))^2 - 4MN$ must be nonpositive (there are no real roots or there is just one), which implies $|f'(x)| \leq 2\sqrt{MN}$. Since x was arbitrarily chosen this bound holds for every $x \in \mathbb{R}$.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, has an intermediate value property, that is if f' takes at least two values $a < b$ then for every $c \in [a, b]$ there exists $x : f'(x) = c$

Solution

Suppose $a = f'(x_1) < c < b = f'(x_2)$. Without loss of generality suppose $x_1 < x_2$. Then it suffices to show that there exists $z \in (x_1, x_2)$ such that $f'(z) = c$.
For $h : 0 < h < x_2 - x_1$, define the secant line by

$$S(x,h) = \frac{f(x+h) - f(x)}{h}$$

Since f is differentiable at each x, note that for each x, $S(x,h) \to f'(x)$ as $h \to 0$ and $S(x-h,h) \to f'(x)$ as $h \to 0$. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $h \in (0, \delta)$,

$$|S(x_1,h) - f'(x_1)| < \epsilon$$

and

$$|S(x_2 - h, h) - f'(x_2)| < \epsilon$$

Thus there exists $\bar{h}$ sufficiently small such that

$$S(x_1, \bar{h}) < c < S(x_2 - \bar{h}, \bar{h})$$

Since f is differentiable it is continuous, which implies $S(\cdot, \bar{h})$ is continuous. By the Intermediate Value Theorem there exists $x \in (x_1, x_2 - \bar{h})$ such that $S(x, \bar{h}) = c$. Then by the Mean Value Theorem there exists $z \in (x, x + \bar{h})$ such that $f'(z) = c$.

2 Implicit and Inverse Function Theorems

Problem 4

The inverse function theorem and implicit function theorem are equivalent theorems, meaning one can be proved using another. In the lecture, you used inverse function theorem to prove implicit function theorem. This problem asks you to prove inverse function theorem using implicit function theorem.
Inverse function theorem:
Suppose $X \subset \mathbb{R}^n$ is open, $f : X \to \mathbb{R}^n$ is C^k on X, and $x_0 \in X$. If $\det Df(x_0) \neq 0$ (i.e. x_0 is a regular point of f) then there are open neighborhoods U of x_0 and V of $f(x_0)$ such that

- $f : U \to V$ is one-to-one and onto
- $f^{-1} : V \to U$ is C^k
- $Df^{-1}(f(x_0)) = [Df(x_0)]^{-1}$

Useful definition: for a function $f : U \to W$, g is right inverse iff $f \circ g$ is the identity map on W. h is left inverse for f iff $h \circ f$ is the identity map on U. f has an inverse (equivalently, is one-to-one and onto) iff $g = h$.

Solution

We have $f : X \to \mathbb{R}^n$, where X is an open subset of $\mathbb{R}^n$, and at $x_0 \in X$, $Df(x_0)$ is invertible.
Set
$$F(x, y) = f(x) - y$$
and
$$y_0 = f(x_0)$$
Note that $F : X \times R^n$, where $X \subset \mathbb{R}^n$. Clearly F is C^k, $F(x_0, y_0) = 0$, and the derivative of F with respect to x at (x_0, y_0) is $D_xF(x_0, y_0) = Df(x_0)$.
Since $Df(x_0)$ is invertible, $D_xF(x_0, y_0)$ is also invertible, and we can apply the Implicit Function Theorem to find neighborhoods $U_{x_0} \subset \mathbb{R}^n$ of x_0 and $W_{y_0} \subset \mathbb{R}^n$ of y_0 such that
$$\forall y \in W_{y_0}, \exists! x \in U_{x_0} : F(x, y) = 0$$
For each y, let $g(y)$ be that unique x. We know that $g : W_{y_0} \to U_{x_0}$ is C^k and it is uniquely defined by the equation
$$F(x, y) = F(g(y), y) = f(g(y)) - y = 0$$
This means that g is "local right inverse" for f in the sense that $f \circ g$ is the identity map on W_{y_0}.
We also get from the Implicit Function Theorem that
$$Dg(y_0) = -[D_xF(x_0, y_0)]^{-1}[D_yF(x_0, y_0)] = [D_xF(x_0, y_0)]^{-1} = [Df(x_0)]^{-1}$$
Loosely speaking, to finish the proof, we show that g is also "local left inverse" for f.
Apply the same analysis with g in place of f since it is also C^k and its derivative at y_0 is invertible, with inverse being $Df(x_0)$. Consequently g has a unique local right inverse on neighborhoods $W'_{y_0} \subseteq W_{y_0}, U'_{x_0} \subseteq U_{x_0}$. Let's call it $h : U'_{x_0} \to W'_{y_0}$. It satisfies $g \circ h = id$ on U'_{x_0}.
Define g' to be restriction of g to W'_{y_0}. Observe that since $f \circ g$ is the identity map on W_{y_0} and $W'_{y_0} \subseteq W_{y_0}$, $(f \circ g')$ is the identity map on W'_{y_0}, implying that g' is "local right inverse" for f. Also note that $g(W'_{y_0}) = U'_{x_0}$, so $g' : W'_{y_0} \to U'_{x_0}$.
On U'_{x_0}, we get $f = f \circ (g' \circ h) = (f \circ g') \circ h = h$. Thus $g' \circ f = g' \circ h = id$ on U'_{x_0} shows that g is "local left inverse" for f and we have $g = f^{-1}$ on a neighborhood of y_0.

Problem 5

Consider the following equations:
$$\begin{aligned} x^2 - yu &= 0, \\ xy + uv &= 0, \end{aligned}$$
where $(x, y, u, v) \in \mathbb{R}^4$. Using the implicit function theorem, describe under what condition these equations can be solved for u and v. Then solve the equations directly and check these conditions.

Solution

Define $f_1 : \mathbb{R}^4 \to \mathbb{R}$ by $f_1(x, y, u, v) = x^2 - yu$ and $f_2 : \mathbb{R}^4 \to \mathbb{R}$ by $f_1(x, y, u, v) = xy + uv$. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $f = (f_1, f_2)$. f is clearly a smooth function. Moreover, the matrix
$$\begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \end{bmatrix} = \begin{bmatrix} -y & 0 \\ v & u \end{bmatrix}$$

has determinant $-yu$.

To determine the suitability of the implicit function theorem, we have to evaluate the determinant at a point (x_0, y_0, u_0, v_0) such that $f_1(x_0, y_0, u_0, v_0) = 0$ and $f_2(x_0, y_0, u_0, v_0) = 0$. Note that $f_1(x_0, y_0, u_0, v_0) = 0$ implies $x_0^2 = y_0 u_0$, so the determinant is equal to $-x_0^2$. Then, the determinant is 0 if and only if $x_0 = 0$. Then, if (x_0, y_0) is such that $x_0 \neq 0$, the conditions of the implicit function theorem are satisfied. This means that there are neighborhoods A of (x_0, y_0) and B of (u_0, v_0) and a unique continuously differentiable function $g : A \to B$ such that $f(x, y, g(x, y)) = 0$ for all $(x, y) \in A$. If we let $u = g_1$ and $v = g_2$, where $g = (g_1, g_2)$, then u and v are the solutions to the system of equations. Then, these equations can be solved uniquely for u and v in neighborhoods around (x_0, y_0) and (u_0, v_0) if $x_0 \neq 0$ (note this also implies that $y_0 \neq 0$).

Solving the system directly gives $u = x^2/y$ and $v = -y^2/x$. Note that these functions are well-defined only when when x and y are different from 0.