Econ 204 – Problem Set 5

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1 Taylor Theorem and Mean Value Theorem

Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Suppose $f'(0) = f'(1) = 2$ and $\forall x \in [0, 1]$, $|f''(x)| \leq 4$.

a. Prove that $|f(1) - f(0)| \leq 4$.

b. Prove that $|f(1) - f(0)| \leq 3$.

(Hint: use Taylor's Theorem)

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 (twice continuously differentiable) function. The function and its second derivative are bounded, namely there exist $M, N > 0$ such that $\sup_{x\in\mathbb{R}} |f(x)| \leq M$ and $\sup_{x\in\mathbb{R}} |f''(x)| \leq N$. Show that $\sup_{x\in\mathbb{R}} |f'(x)| \leq 2\sqrt{MN}$.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, has an intermediate value property, that is if f' takes at least two values $a < b$ then for every $c \in [a, b]$ there exists $x : f'(x) = c$

2 Implicit and Inverse Function Theorems

Problem 4

The inverse function theorem and implicit function theorem are equivalent theorems, meaning one can be proved using another. In the lecture, you used inverse function theorem to prove implicit function theorem. This problem asks you to prove inverse function theorem using implicit function theorem.
Inverse function theorem:
Suppose $X \subset \mathbb{R}^n$ is open, $f : X \to \mathbb{R}^n$ is C^k on X, and $x_0 \in X$. If $\det Df(x_0) \neq 0$ (i.e. x_0 is a regular point of f) then there are open neighborhoods U of x_0 and V of $f(x_0)$ such that

- $f : U \to V$ is one-to-one and onto

- $f^{-1} : V \to U$ is C^k
- $Df^{-1}(f(x_0)) = [Df(x_0)]^{-1}$

Useful definition: for a function $f : U \to W$, g is right inverse iff $f \circ g$ is the identity map on W. h is left inverse for f iff $h \circ f$ is the identity map on U. f has an inverse (equivalently, is one-to-one and onto) iff $g = h$.

Problem 5

Consider the following equations:

$$\begin{aligned} x^2 - yu &= 0, \\ xy + uv &= 0, \end{aligned}$$

where $(x, y, u, v) \in \mathbb{R}^4$. Using the implicit function theorem, describe under what condition these equations can be solved for u and v. Then solve the equations directly and check these conditions.