Econ 204 – Problem Set 6 Suggested Solutions

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1 Fixed points

Problem 1

Suppose $\Psi : X \to 2^X$ is a non-empty and compact-valued upper-hemicontinuous correspondence. The metric space X is compact. Show that there exists a non-empty compact set $C \subset X$ such that $\Psi(C) = C$.
Hint: for one direction, use the result that you proved in HW3 that the image of every compact subset under such a correspondence is compact.

Solution

First recall that the image of every compact subset under such a correspondence is compact. Therefore, $\Psi(X)$ is compact and $\Psi(X) \subset X$. Hence, $\Psi^2(X) := \Psi(\Psi(X)) \subset \Psi(X)$ is also compact. Consequently, we can construct a decreasing sequence of compact subsets $\{\Psi^n(X)\}$ such that $\Psi^{n+1}(X) := \Psi(\Psi^n(X))$ and $\Psi^n(X) \supset \Psi^{n+1}(X) \supset \dots$. Let $C = \bigcap_{n\in\mathbb{N}} \Psi^n(X)$, which is non-empty because of Cantor theorem, and is closed because it is the intersection of closed subsets. So, C is compact as it is a closed subset of a compact set X. Since $C \subset \Psi^n(X)$ for every n, then $\Psi(C) \subset \Psi(\Psi^n(X)) = \Psi^{n+1}(X)$, and hence $\Psi(C) \subset \bigcap_{n\in\mathbb{N}} \Psi^n(X) = C$. Thus it is enough to show $C \subset \Psi(C)$. For this we offer two proofs; the first one is based on the sequential characterization of uhc and the second one uses the open set definition.

First proof: Let $y \in C$. By definition $y \in \Psi^n(X)$ for every n, so for every n there exists $z_n \in \Psi^{n-1}(X)$ such that $y \in \Psi(z_n)$. Then $\{z_n\} \subseteq X$ and X is compact, so there is a convergent subsequence $z_{n_k} \to z \in X$. Since $y \in \Psi(z_{n_k})$ for each n_k and Ψ has closed graph (because Ψ is uhc and closed-valued), must have $y \in \Psi(z)$ as well. Now claim $z \in C$. If not, then there exists N such that for all $n \geq N$, $z \notin \Psi^n(X)$. In particular, $z \notin \Psi^N(X)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \cap \Psi^N(X) = \emptyset$. But this is a contradiction, as $z_{n_k} \in \Psi^N(X)$ for all $n_k > N$, and $z_{n_k} \to z$. Therefore $z \in C$. Thus $y \in \Psi(C)$. So $C \subset \Psi(C)$.

Second proof: Assume to the contrary, $\exists z \in C \setminus \Psi(C)$. Therefore, $\{z\}$ and $\Psi(C)$ are disjoint. Since $\Psi(C)$ is closed one can find an open ball $B(z, \varepsilon_z)$ around z such that $B(z, \varepsilon_z) \cap \Psi(C) = \emptyset$. Let $\bar{B}(z, \varepsilon_z/2) = \{y \in X : d(y, z) \leq \varepsilon/2\}$ be the closed ball with radius $\varepsilon_z/2$ around z, then $\bar{B}(z, \varepsilon_z/2) \subset B(z, \varepsilon_z)$. Now for every point $x \in \Psi(C)$ one can find an open ball $B(x, \varepsilon_x)$ such that $B(x, \varepsilon_x) \cap \bar{B}(z, \varepsilon_z/2) = \emptyset$. Let G be the union of all these balls, i.e $G = \bigcup_{x\in\Psi(C)} B(x, \varepsilon_x)$, then G

is an open set containing $\Psi(C)$ that is disjoint from $B(z, \varepsilon_z/2)$ containing z. Because of uhc the upper-inverse $\Psi^u(G)$ is open and covers C. There must be some $N \in \mathbb{N}$ such that for all $n \geq N$, $\Psi^n(X) \subset \Psi^u(G)$, because otherwise one could employ an elementary compactness argument to reach a contradiction. This implies $\Psi^{N+1}(X) = \Psi(\Psi^N(X)) \subset G$, so $C \subset G$, that violates the disjointness of $z \in C$ from G. Therefore, $C \subset \Psi(C)$.

Problem 2

a) *Berge's Maximum Theorem:* Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Consider the function $f : X \times Y \to \mathbb{R}$ and the correspondence $\Gamma : Y \twoheadrightarrow X$. Define $v(y) = \max_{x \in \Gamma(y)} f(x, y)$ and $\Omega(y) = \arg\max_{x \in \Gamma(y)} f(x, y)$. Suppose f and Γ are continuous, and that Γ has non-empty compact values. Show that v is continuous and Ω is uhc with non-empty compact values.
Hint: you may find useful to use the sequential definitions of uhc and lhc.

b) Assume that Γ also has convex values. Show that if f is quasi-concave in x, Ω has convex values.[1]

c) Let $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ denote a *social game*, where I is the (finite) set of players, and $u^i : \prod_{j \in I} S^j \to \mathbb{R}$ is the objective function of player $i \in I$ defined over $s = \left(s^j; j \in I\right) \in \prod_{j \in I} S^j$, with $S^j \subset \mathbb{R}^{n_j}$, $n_j > 0$. Each player i chooses $s^i \in \arg\max_{s \in \Gamma^i(s_{-i})} u^i(s, s_{-i})$, with $s_{-i} := (s_j; j \in I \setminus \{i\})$, and $\Gamma^i(s_{-i}) \subset S^i$. Define an equilibrium for the social game $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ as a vector $\bar{s} = \left(\bar{s}^i; i \in I\right)$ such that, $\forall i \in I$, $u^i(\bar{s}) \geq u^i(s, \bar{s}_{-i})$, $\forall s \in \Gamma^i\left(\bar{s}_{-i}\right)$, where $\bar{s}_{-i} := \left(\bar{s}^j; j \neq i\right)$. Assume S^i is convex, compact, and non-empty for each $i \in I$, and that u^i is continuous and quasi-concave in s^i for each $i \in I$. Use the previous parts of this question to show that, if $\{\Gamma^i\}_{i \in I}$ are continuous and have compact, convex, and non-empty values, then an equilibrium for $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ exists.

Solution

a) Since Γ has non-empty compact values, continuity of f implies that, $\forall y \in Y$, $\Omega(y) \neq \emptyset$. Given $y \in Y$, let $\{x_n\}_{n \in \mathbb{N}} \in \Omega(y)$ be a sequence that converges to $x \in X$. Fix $x' \in \Gamma(y)$. Since $x_n \in \Omega(y)$ for each n, $f(x_n, y) \geq f(x', y)$ for each n. Since $x_n \to x$ and f is continuous, this implies $f(x, y) \geq f(x', y)$. Since $x' \in \Gamma(y)$ was arbitrary, this implies that $x \in \Omega(y)$. Thus $\Omega(y)$ is closed. Since $\Omega(y) \subset \Gamma(y)$ and $\Gamma(y)$ is compact, $\Omega(y)$ is compact.

To show that Ω is uhc, let's use the sequential characterization (that can be used because Ω has compact values). Fix $y \in Y$ and take a sequence $\{y_n\}_{n \in \mathbb{N}}$ that converges to $y \in X$. Given a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that $x_n \in \Omega(y_n)$, $\forall n \in \mathbb{N}$, we know that $x_n \in \Gamma(y_n)$, $\forall n \in \mathbb{N}$. Then, given that Γ is uhc, we know that exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ that converges to some $x \in \Gamma(y)$. So it suffices to show that $x \in \Omega(y)$. To that end, let $x' \in \Gamma(y)$. Since Γ is lhc, there exists a sequence $x'_n \to x$ such that $x'_n \in \Gamma(y_n)$ for each n. Then for each k, $f(x_{n_k}, y_{n_k}) \geq f(x'_{n_k}, y_{n_k})$. Letting $k \to \infty$, this implies $f(x, y) \geq f(x', y)$. Since $x' \in \Gamma(y)$ was arbitrary, this implies $x \in \Omega(y)$. Thus Ω is uhc.

[1] A function $f : X \to \mathbb{R}$ is quasi-concave if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $f(\lambda x_1 + (1 - \lambda) x_2) \leq \max\{f(x_1), f(x_2)\}$.

Finally, to show that v is continuous, consider a sequence $\{y_n\} \subset Y$ that converges to $y \in Y$. We know that, $\forall n \in \mathbb{N}$, exists $x_n \in \Gamma(y_n)$ such that $v(y_n) = f(x_n, y_n)$. That is, exists $x_n \in \Omega(y_n)$. Since Ω is uhc, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to some point $x \in \Omega(y)$. Likewise, since Ω is lhc, there exists a sequence $\{\tilde{x}_n\}$ such that $\tilde{x}_n \to x$ and $\tilde{x}_n \in \Omega(y_n)$ for each n. Since f is continuous, $v(y_n) = f(\tilde{x}_n, y_n) \to f(x, y) = v(y)$. Thus v is continuous.

b) Given $y \in Y$, take two points x_1 and x_2 in $\Omega(y)$. Since $\Gamma(y)$ has convex values, $\forall\lambda \in (0, 1)$, $z_\lambda := \lambda x_1 + (1-\lambda)x_2 \in \Gamma(y)$. Also, since f is quasi-concave in x, $f(z_\lambda, y) \geq \min\{f(x_1, y), f(x_2, y)\}$. Then, $z_\lambda \in \Omega(y)$, $\forall\lambda \in (0, 1)$. Then, Ω has convex values.

c) Identify the set I with $\{1, ..., \kappa\}$ for some $k \in \mathbb{N}$ (which we can do without loss of generality since I is finite). Define, for all $i \in I$, the correspondence $\Phi^i : \prod_{j\neq i} S^j \twoheadrightarrow S^i$ as $\Phi^i(s_{-i}) := \arg\max_{s\in\Gamma^i(s_{-i})} u^i(s, s_{-i})$. Given Berge's Maximum Theorem, we know that, $\forall i \in I$, Φ^i is uhc and has compact, convex, and non-empty values. Then, the correspondence $\Phi : \prod_{i=1}^{\kappa} S^i \twoheadrightarrow \prod_{i=1}^{\kappa} S^i$ defined by $\Phi(s) = \Phi^1(s_{-1}) \times ... \times \Phi^\kappa(s_{-\kappa})$ satisfies the hypotheses of Kakutani's fixed point theorem. Then, there is $\bar{s} = \left(\bar{s}^i; i \in I\right)$ such that $\bar{s} \in \Phi(\bar{s})$. This fixed point $\bar{s}$ is an equilibrium for the social game.

2 Separating Hyperplane Theorem

Problem 3

1. Let A and B be disjoint nonempty convex subsets of $\mathbb{R}^n$ and suppose $p \in R^n$ is a non-zero vector that separates A and B with $p \cdot a \geq p \cdot b$ for all $a \in A, b \in B$. If A includes a set of the form $\{x\} + \mathbb{R}^n_{++}$, then $p > 0$.
   Hint: proof by contradiction.

2. Let C be a nonempty convex subset of a vector space, and let $f_1, \ldots, f_m : C \to \mathbb{R}$ be concave. Letting $f = (f_1, \ldots, f_m) : C \to \mathbb{R}^m$, exactly one of the following is true:

   a
   $$\exists\bar{x} \in C \text{ such that } f(\bar{x}) > 0$$

   b
   $$\exists p > 0 \text{ such that } p \cdot f(x) \leq 0 \text{ for all } x \in C$$

Solution

1. Since p is nonzero by hypothesis, it suffices to show that $p \geq 0$. Suppose by way of contradiction that $p_i < 0$ for some i. Fix $y \in B$. Note that $t\mathbf{e_i} + \mathbf{1}$ belongs to $\mathbb{R}^n_{++}$ for every $t > 0$. Now $p \cdot (x + t\mathbf{e_i} + \mathbf{1}) = p \cdot x + tp_i + p \cdot \mathbf{1}$. By letting $t \to +\infty$, we see that $p \cdot x + tp_i + p \cdot \mathbf{1} \to -\infty$, which contradicts $p \cdot (x + t\mathbf{e_i} + \mathbf{1}) \geq p \cdot y$ for any $t > 0$. Therefore $p > 0$.

2. This is known as Concave Alternative Theorem.
   Clearly both cannot be true. Suppose (a) fails. Set $H = f(C)$ and set $\hat{H} = \{y \in \mathbb{R}^m : y \leq f(x)$ for some $x \in C\}$. Since (a) fails, we see that H and $\mathbb{R}^m_{++}$ are disjoint. Consequently $\hat{H}$ and $\mathbb{R}^m_{++}$ are disjoint.

Now observe that $\hat{H}$ is convex. To see this, take $y_1, y_2 \in \hat{H}$. Then there exists x_1, x_2 such that $y_i \leq f(x_i), i = 1, 2$. Therefore, for any $\lambda \in (0, 1)$,

$$\lambda y_1 + (1 - \lambda) y_2 \leq \lambda f(x_1) + (1 - \lambda) f(x_2) \leq f(\lambda x_1 + (1 - \lambda) x_2)$$

since each f_j is concave. We arrive at $\lambda y_1 + (1 - \lambda) y_2 \in \hat{H}$.
Since $\hat{H}$ is convex, by the Separating Hyperplane Theorem, there is a nonzero vector $p \in \mathbb{R}^m$ separating $\hat{H}$ and $\mathbb{R}^m_{++}$. We may assume p is such that

$$p \cdot y \leq p \cdot r \text{ for all } y \in \hat{H}, r \in \mathbb{R}^m_{++}$$

By (1), $p > 0$. Observe that for any $\epsilon > 0$, $\epsilon \mathbf{1} \in \mathbb{R}^m_{++}$. Therefore, $p \cdot y \leq \epsilon p \cdot \mathbf{1} \ \forall y \in \hat{H}$. Since ϵ may be taken arbitrarily small, we conclude that $p \cdot y \leq 0 \ \forall y \ \in \hat{H}$. In particular, $p \cdot f(x) \leq 0 \ \ \forall x \ \in C$.

3 Differential equations

Problem 4

Solve the following differential equation: $y'' - 5y' + 4y = e^{4x}$. Concretely, provide (i) the general solution of the homogeneous differential equation, and (ii) the particular and general solutions of the inhomogeneous differential equation. Solve explicitly for the constants using the following initial conditions: $y(0) = 3$, $y(0)' = \frac{19}{3}$.

Solution

The characteristic equation is $k^2 - 5k + 4 = 0$. The solutions are given by $k_1 = 4$ and $k_2 = 1$. Then, the general solution of the homogeneous equations is $y_0(x) = C_1 e^{4x} + C_2 e^x$.

For the particular inhomogeneous solution, we conjecture that $y_1(x) = xAe^{4x}$. Then

$$\begin{aligned} y_1'(x) &= (A + 4Ax)e^{4x}, \\ y_2''(x) &= (8A + 16Ax)e^{4x}. \end{aligned}$$

This implies that $(8A + 16Ax)e^{4x} - 5(A + 4Ax)e^{4x} + 4xAe^{4x} = e^{4x}$. Solving for A yields $A = \frac{1}{3}$ and, therefore, $y_1(x) = \frac{x}{3} e^{4x}$.

The general inhomogeneous solution is $y_0(x) + y_1(x) = C_1 e^{4x} + C_2 e^x + \frac{x}{3} e^{4x}$.

Finally, using the initial conditions, we have that $C_1 + C_2 = 3$ and $4C_1 + C_2 + \frac{1}{3} = \frac{19}{3}$. Then, $C_1 = 1$ and $C_2 = 2$.