Econ 204 – Problem Set 6

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1 Fixed points

Problem 1

Suppose $\Psi : X \to 2^X$ is a non-empty and compact-valued upper-hemicontinuous correspondence. The metric space X is compact. Show that there exists a non-empty compact set $C \subset X$ such that $\Psi(C) = C$.
Hint: for one direction, use the result that you proved in HW3 that the image of every compact subset under such a correspondence is compact.

Problem 2

a) *Berge's Maximum Theorem:* Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Consider the function $f : X \times Y \to \mathbb{R}$ and the correspondence $\Gamma : Y \twoheadrightarrow X$. Define $v(y) = \max_{x \in \Gamma(y)} f(x, y)$ and $\Omega(y) = \arg\max_{x \in \Gamma(y)} f(x, y)$. Suppose f and Γ are continuous, and that Γ has non-empty compact values. Show that v is continuous and Ω is uhc with non-empty compact values.
Hint: you may find useful to use the sequential definitions of uhc and lhc.

b) Assume that Γ also has convex values. Show that if f is quasi-concave in x, Ω has convex values.[1]

c) Let $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ denote a *social game*, where I is the (finite) set of players, and $u^i : \prod_{j \in I} S^j \to \mathbb{R}$ is the objective function of player $i \in I$ defined over $s = \left(s^j; j \in I\right) \in \prod_{j \in I} S^j$, with $S^j \subset \mathbb{R}^{n_j}$, $n_j > 0$. Each player i chooses $s^i \in \arg\max_{s \in \Gamma^i(s_{-i})} u^i(s, s_{-i})$, with $s_{-i} := (s_j; j \in I \setminus \{i\})$, and $\Gamma^i(s_{-i}) \subset S^i$. Define an equilibrium for the social game $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ as a vector $\bar{s} = \left(\bar{s}^i; i \in I\right)$ such that, $\forall i \in I$, $u^i(\bar{s}) \geq u^i(s, \bar{s}_{-i})$, $\forall s \in \Gamma^i\left(\bar{s}_{-i}\right)$, where $\bar{s}_{-i} := \left(\bar{s}^j; j \neq i\right)$. Assume S^i is convex, compact, and non-empty for each $i \in I$, and that u^i is continuous and quasi-concave in s^i for each $i \in I$. Use the previous parts of this question to show that, if $\{\Gamma^i\}_{i \in I}$ are continuous and have compact, convex, and non-empty values, then an equilibrium for $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ exists.

[1] A function $f : X \to \mathbb{R}$ is quasi-concave if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $f(\lambda x_1 + (1-\lambda) x_2) \leq \max\{f(x_1), f(x_2)\}$.

2 Separating Hyperplane Theorem

Problem 3

1. Let A and B be disjoint nonempty convex subsets of $\mathbb{R}^n$ and suppose $p \in R^n$ is a non-zero vector that separates A and B with $p \cdot a \geq p \cdot b$ for all $a \in A, b \in B$. If A includes a set of the form $\{x\} + \mathbb{R}^n_{++}$, then $p > 0$.
   Hint: proof by contradiction.

2. Let C be a nonempty convex subset of a vector space, and let $f_1, \ldots, f_m : C \to \mathbb{R}$ be concave. Letting $f = (f_1, \ldots, f_m) : C \to \mathbb{R}^m$, exactly one of the following is true:

   a
   $$\exists \bar{x} \in C \text{ such that } f(\bar{x}) > 0$$

   b
   $$\exists p > 0 \text{ such that } p \cdot f(x) \leq 0 \text{ for all } x \in C$$

3 Differential equations

Problem 4

Solve the following differential equation: $y'' - 5y' + 4y = e^{4x}$. Concretely, provide (i) the general solution of the homogeneous differential equation, and (ii) the particular and general solutions of the inhomogeneous differential equation. Solve explicitly for the constants using the following initial conditions: $y(0) = 3$, $y(0)' = \frac{19}{3}$.