Sharing Risk and Ambiguity[*]

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Abstract

We study the market implications of ambiguity in many common models. We show that generic determinacy is a robust feature in many general equilibrium models that allow a distinction between ambiguity and risk.

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1 Introduction

Ambiguity describes situations in which probabilities are defined only imprecisely. For example, imagine a box contains 100 black and white balls. A decision maker knows only that at least 20 and no more than 50 of these balls are black, and must choose between bets that depend on the color of a ball drawn randomly from this box. A classic illustration of possible behavioral consequences of this situation is due to Ellsberg (1961). Agents may prefer bets contingent on events with unique probabilities to those with unknown probabilities, in an illustration of a phenomenon sometimes called ambiguity aversion. Such behavior is inconsistent not only with expected utility maximization, but with any model of probabilistically sophisticated choice in which choice is based on a unique subjective prior over the state space. Many alternatives to expected utility have been developed over the last twenty years to capture ambiguity-averse behavior.

We study the market implications of the most prevalent of these models, and address the question of whether ambiguity plays an important role in market outcomes. In the standard Arrow-Debreu setting with expected utility, subjective probabilities directly affect market prices via individuals' marginal rates of substitution for state contingent commodity bundles. In the simplest version of this model, equilibrium price vectors are given by marginal utility weighted probabilities (also referred to as "risk-neutral probabilities"). Since ambiguity typically involves multiple subjective probabilities, it may be reflected in multiplicity of equilibrium prices. This has been one of the most important market implications of ambiguity, first illustrated by Dow and Werlang (1992) in a partial equilibrium portfolio choice setting, and by Epstein and Wang (1994) in a version of Lucas' asset pricing model, using Choquet expected utility.[1] In this spirit, models incorporating ambiguity have recently been advanced as explanations for a wide variety of market anomalies, including bid-ask spreads, excess price volatility, home bias, indeterminacies in equilibrium, and inertia in trading.[2]

We show that if ambiguity is modeled using preferences drawn from many standard classes that display ambiguity aversion, patterns of risk and ambiguity sharing arising in equilibrium cannot robustly be distinguished from those arising in a standard expected utility model. In particular, equilibrium prices will be generically determined. We adopt the framework of variational preferences recently axiomatized by Maccheroni, Marinacci, and Rustichini (2006), which provides a convenient way to nest many models of ambiguity while allowing heterogeneity in the way ambiguity is expressed. These preferences include as special cases those primarily

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[1]Similar results can be found in Anderson, Hansen, and Sargent (2001) and Hansen and Sargent (2001).

[2]Some notable examples include Anderson, Hansen, and Sargent (2001), Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000), Dana (2002) and Dana (2004), Dow and Werlang (1992), Epstein (2001), Epstein and Wang (1994), Hansen, Sargent, and Tallarini (1999), Kajii and Ui (2006), Maenhout (2004), Mukerji and Tallon (2001), and Tallon (1998). For two overviews see Backus, Routledge, and Zin (2004) and Mukerji and Tallon (2004).

used to study the market implications of ambiguity aversion, including the maxmin expected utility preferences of Gilboa and Schmeidler (1989), convex Choquet expected utility preferences axiomatized by Schmeidler (1989), the multiplier preferences proposed by Hansen and Sargent (see Hansen and Sargent (2006) for an overview), and mean-variance preferences. We show that if agents have variational preferences, and hence any combination of preferences drawn from these classes, equilibria are generically determinate and trade typically occurs.

At first glance, these results may seem at odds with an array of recent papers suggesting that ambiguity aversion can have important implications for market prices and allocations. Perhaps the first paper to emphasize this point was Dow and Werlang (1992) in the context of a portfolio choice problem, using convex Choquet expected utility preferences. Epstein and Wang (1994) extend this partial equilibrium indeterminacy result to a representative agent asset pricing model with Choquet expected utility. In these papers, the perception of ambiguity is typically reflected through kinks in otherwise standard complete, convex preferences. These kinks occur where the set of effective priors is multi-valued, for example where there are many distributions minimizing the expected value of state-contingent utility for maxmin preferences. Many market anomalies attributed to ambiguity derive from such kinks, including bid-ask spreads, indeterminacies in equilibrium, and inertia in trading.

We show in contrast that indeterminacies and no trade are not typical equilibrium phenomena. A partial intuition for this result comes from thinking about a representative agent model. In such a model, equilibrium consists of the initial endowment bundle, together with prices that support the agent's consumption of this bundle. Indeterminacies arise whenever the initial endowment coincides with a kink in the agent's preferred set. Because variational preferences have a concave representation, at most a measure zero set of bundles could generate such kinks, so equilibrium indeterminacies are absent for almost all choices of initial endowment. The fact that there is no trade in equilibrium provides an immediate link between kinks in preferred sets and indeterminacies in equilibrium prices, and renders immediate the conclusion that indeterminacies occur only for a particular, and small, set of endowments.

This intuition is misleading in general, however, precisely because there is never trade in equilibrium with a representative agent. With multiple agents, markets provide opportunities for trade, and heterogeneity in endowments or preferences typically produces incentives to trade. Equilibrium allocations are then determined endogenously, as an outcome of these forces, which might result in trade occuring at consumption bundles where preferences have kinks even if the set of such bundles is small. Another important special case, the absence of aggregate uncertainty, illustrates this point and suggests an opposite intuition from the representative agent case. With no aggregate uncertainty, the only Pareto efficient allocations are those involving full insurance under many classes of preferences, provided agents share at least one common prior (see Billot, Chateauneuf, Gilboa, and Tallon (2000), Dana (2002), and Rigotti, Shannon, and Strzalecki (2007)). Since equilibrium allocations must be efficient, they must also involve full insurance regardless of agents' initial endowments. In this setting equilibrium

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involves bundles at which kinks occur for any agent who does not have a unique prior. Despite the rarity of these bundles, trade always brings equilibrium to such allocations, regardless of initial endowments. The link between kinks and indeterminacies is further complicated by the endogeneity of equilibrium prices with heterogeneous agents.

Our results generalize earlier results of Chateauneuf, Dana, and Tallon (2000) and Dana (2004). These papers consider models in which agents with Choquet expected utility preferences share a common convex capacity. These preferences become a special case of maxmin expected utility in which agents share a particular set of priors, the core of this common convex capacity. Chateauneuf, Dana, and Tallon (2000) shows that efficient allocations are comonotonic in this case: efficiency requires that all individual consumption vectors agree on the ranking of states from "best" to "worst" in terms of ex-post consumption. When aggregate uncertainty is present, the set of comonotonic allocations will typically have a non-empty interior, and agents' preferences will typically coincide with standard expected utility in a neighborhood of every efficient allocation. Building on this result, Dana (2004) shows that there are no price indeterminacies in an economy in which aggregate endowments are a one-to-one mapping of states and agents have a common convex capacity. Equilibria are thus generically determinate in this case, and coincide with equilibria in a standard expected utility model with fixed priors.

We show that these basic features of equilibrium allocations and prices hold more generally, although by necessity our techniques are very different. The results of Chateauneuf, Dana, and Tallon (2000) and Dana (2004) are difficult to extend beyond the common convex capacity case as they crucially depend on characteristics of the set of probabilities minimizing expected utility for each agent. This set has a particularly simple structure in the case of Choquet expected utility, and the problem is further simplified by their assumption that agents share a common such set of beliefs. Outside of the case of a common convex capacity, the only known results link price indeterminacies to the absence of aggregate uncertainty (see Dana (2004)). A main contribution of our paper is to provide results and techniques that go beyond these special cases. Because many preferences with ambiguity are not smooth, we cannot study equilibria in these markets using the traditional techniques of differential topology pioneered by Debreu (1970). In the context of uncertainty, failures of differentiability have important behavioral content, since they are a direct reflection of individual ambiguity aversion, and are not merely a technicality in modeling. Instead, we use a variety of alternative techniques that do not rely on differentiability, such as those developed by Blume and Zame (1993), Pascoa and Werlang (1999), and Shannon (1994).

The remainder of this paper proceeds as follows. The next section describes the model. Section 3 presents the results on determinacy. Throughout we discuss the extent to which our results also encompass other models of ambiguity not nested within the variational preferences framework.

2 The Model

In this section, we describe a simple exchange economy in which agents' preferences over contingent consumption allow for the perception of ambiguity. Aside from this feature, this framework is the standard Arrow-Debreu model of complete contingent security markets.

There are two dates, 0 and 1. At date 0, there is uncertainty about which state s from a state space S will occur at date 1. The state space S is finite, and we abuse notation by letting S denote the number of states as well. Let $\Delta := \{\pi \in \mathbf{R}_+^S : \sum \pi_s = 1\}$ denote the probability simplex in $\mathbf{R}_+^S$. A single good is available for consumption at date 1 and, for simplicity, we assume no consumption takes place before then. At date 0 agents can trade in a complete set of Arrow securities for contingent consumption at date 1, hence have consumption set $\mathbf{R}_+^S$.

There are finitely many agents, indexed by $i = 1, \ldots, m$. Each agent has an endowment $\omega^i \in \mathbf{R}_{++}^S$ of contingent consumption at date 1, and a preference order $\succeq^i$ over $\mathbf{R}_+^S$. We will maintain familiar basic assumptions on preferences, including completeness, continuity, and convexity.

Standard applied analyses of risk sharing in this context typically assume that preferences have an expected utility representation. Instead, standard general equilibrium analyses allow for arbitrary convex preference orders, which accommodate a range of models incorporating perceptions of ambiguity. For most of our results, we assume that each agent's preference order has a variational representation, as axiomatized by Maccheroni, Marinacci, and Rustichini (2006) in an Anscombe-Aumann setting and described in detail below. This class is general enough to nest many of the models developed and applied to study ambiguity, including the maxmin multiple priors preferences of Gilboa and Schmeidler, the convex Choquet expected utility preferences of Schmeidler, and the multiplier preferences used by Hansen and Sargent.[3] At the same time, this class is sufficiently tractable and precise to allow us to link our results to identifiable features of preferences. Below we comment on ways our results can be extended to include preferences that do not have a variational representation.

In our basic model, we maintain the assumption that for each agent $i = 1, \ldots, m$ there exists a utility index $u^i : \mathbf{R}_+ \to \mathbf{R}$ and a convex, lower semi-continuous function $c^i : \Delta \to [0, \infty]$ such that

$$x \succeq^i y \iff \min_{\pi \in \Delta} \left\{ E_\pi[u^i(x)] + c^i(\pi) \right\} \geq \min_{\pi \in \Delta} \left\{ E_\pi[u^i(y)] + c^i(\pi) \right\}$$

where $E_\pi u^i(x) := \sum_s \pi_s u^i(x_s)$. In this framework, agent i's preference order is represented by the utility function $V^i : \mathbf{R}_+^S \to \mathbf{R}$ given by

$$V^i(x) = \min_{\pi \in \Delta} \left\{ E_\pi[u^i(x)] + c^i(\pi) \right\}.$$

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[3]Notable exceptions are the smooth second-order probability models of Ergin and Gul (2004), Klibanoff, Marinacci, and Mukerji (2005) and Nau (2006).

We say that $\succeq^i$ has a *variational representation* if $\succeq^i$ is represented by a utility function V^i of this form.

We will maintain the following assumptions throughout:

(A) for each i:

1. $\succeq^i$ is a complete, transitive order on $\mathbf{R}_+^S$ that is continuous, strictly convex, and strictly monotone
2. $\succeq^i$ has a variational representation such that
   (i) u^i is strictly concave, strictly increasing, and continuously differentiable on $\mathbf{R}_{++}$;
   (ii) c^i is properly grounded , i.e.

$$N^i := \{\pi \in \Delta : c^i(\pi) = 0\} \neq \emptyset$$

The sets N^i will play a particularly central role in characterizing the efficiency of full insurance in the following section. For reasons that will become clear below, we call N^i the set of *state-neutral priors* for agent i.

Assumption A1 is standard, and allows us to appeal to all of the basic general equilibrium results, including the welfare theorems. Assumption A2 strengthens the variational framework slightly to rule out failures of differentiability unrelated to ambiguity.

This model admits as a special case maxmin expected utility introduced by Gilboa and Schmeidler (1989), when c is given by an indicator function of the form

$$c_\Pi(\pi) = \begin{cases} 0 & \text{if } \pi \in \Pi \\ \infty & \text{if } \pi \notin \Pi \end{cases}$$

for a convex, compact set $\Pi \subset \Delta$. In this case,

$$\min_{\pi \in \Delta} \left\{ E_\pi[u^i(x)] + c^i(\pi) \right\} = \min_{\pi \in \Pi} E_\pi[u^i(x)]$$

and $\Pi = \{\pi \in \Delta : c_\Pi(\pi) = 0\}$. This also incorporates convex preferences having a Choquet expected utility representation, which are themselves special cases of maxmin preferences for which the set Π is the core of a convex capacity.

This model also encompasses the multiplier preferences used to model robustness to uncertainty by a number of authors; see Hansen and Sargent (2006) for an overview. Here $c(\pi) = \theta R(\pi || \hat{\pi})$ for some $\theta > 0$ and $\hat{\pi} \in \Delta$, with $\hat{\pi} \gg 0$, where R denotes the relative entropy between π and $\hat{\pi}$ defined by

$$R(\pi || \hat{\pi}) := \sum_{s \in S} \pi_s \log \frac{\pi_s}{\hat{\pi}_s}.$$

Notice that in this case, $c(\pi) = 0$ if and only if $\pi = \hat{\pi}$.

Many preferences in this class are consistent with the types of behavior that have been attributed to ambiguity in some of the recent literature, including bid-ask spreads in equilibrium prices, inertia in trading, and home bias.

A few preliminary lemmas establish some useful features of variational preferences. These features lead to a characterization of efficient allocations in this section, and are helpful in understanding the generic determinacy of equilibrium in the following section.

For each $x \in \mathbf{R}_+^S$ let

$$M^i(x) := \left\{ \pi \in \Pi^i : \pi \in \arg\min_{\pi \in \Delta} \left\{ E_\pi[u^i(x)] + c^i(\pi) \right\} \right\}$$

be the set of minimizing priors realizing the utility of x. Note that $V^i(x) = E_\pi[u^i(x)] + c^i(\pi)$ for each $\pi \in M^i(x)$; accordingly we refer to $M^i(x)$ as the set of *effective priors* at x.

Let $\partial V^i(x)$ denote the set of subgradients of V^i at x. Since V^i is concave, this set is always nonempty, and the set of subgradients characterizes the points of differentiability of V^i through uniqueness. To derive a simple characterization of the set of subgradients, denote by $U^i : \mathbf{R}_+^S \to \mathbf{R}^S$ the function $U^i(x) := (u^i(x_1), \ldots, u^i(x_S))$ giving ex-post utilities in each state. For any $x \in \mathbf{R}_{++}^S$, $DU^i(x)$ is the $S \times S$ diagonal matrix with diagonal $\left(Du^i(x_1), \ldots, Du^i(x_S) \right)$ the vector of ex-post marginal utilities. From the form of the utility V^i and a standard envelope theorem, we can express the set of subgradients of V^i easily using these components.

Lemma 1 *Under Assumption A, for each $x \in \mathbf{R}_{++}^S$,*

$$\partial V^i(x) = \{ q \in \mathbf{R}^S : q = \pi DU^i(x) \text{ for some } \pi \in M^i(x) \}.$$

Proof: This follows from the definitions of V^i, U^i, and M^i, and Clarke (1983, 2.8, Cor. 2). ∎

We also note a useful consequence of strict monotonicity for the set effective priors.

Lemma 2 *Under Assumption A, $M^i(x) \subset \Delta \cap \mathbf{R}_{++}^S$ for each $x \in \mathbf{R}_{++}^S$.*

Proof: Fix $x \in \mathbf{R}_{++}^S$. By strict monotonicity, $V^i(x') > V^i(x)$ for any $x' > x$, and by Lemma 1,

$$0 < V^i(x') - V^i(x) \leq \pi DU^i(x) \cdot (x' - x)$$

for each $\pi \in M^i(x)$ and $x' > x$. Together with the observation that $Du^i(x_s) > 0$ for each s, because u^i is concave and strictly increasing, this inequality establishes that $\pi \gg 0$ for each $\pi \in M^i(x)$. ∎

3 Determinacy and Multiple Priors

In this section, we study the determinacy of equilibrium, and provide two results. These results shed light on the extent to which the various market anomalies associated with ambiguity are robust features of equilibrium. For the general case of variational preferences, we show that generically equilibria are determinate in the sense that there are only finitely many equilibrium prices and allocations and the equilibrium correspondence is continuous at almost all endowment profiles. The intuition for this result is fairly straightforward. With variational preferences, each consumer's utility function is strictly concave; while utility may not be differentiable everywhere, it is nonetheless differentiable almost everywhere as a consequence. Loosely, while the nondifferentiabilities introduce an additional possible source of indeterminacy, realizing these indeterminacies requires equilibrium to occur at allocations where agents share points of nondifferentiability. This is not a robust feature of equilibrium when the kinks are not robust.

When agents' preferences have a Choquet expected utility representation, we use the additional structure this imposes on agents' sets of effective priors to derive a slightly stronger result. In this case, in addition to generic determinacy, we derive a slightly stronger result concerning equilibrium comparative statics.

For each of these results we assume that agents' preferences satisfy the following versions of standard differential convexity and Inada conditions:

(D) $u^i : \mathbf{R}_+ \to \mathbf{R}$ is C^2; $D^2 u^i(y) < 0$ for each $y > 0$; and $Du^i(y) \to \infty$ as $y \to 0$ for each i

We let $\omega = (\omega^1, \ldots, \omega^m)$ denote the profile of individual endowments, and $\mathcal{E}(\omega)$ denote the economy with primitives $\{(\succeq^i, \omega^i) : i = 1, \ldots, m\}$. We normalize prices $p \in \mathbf{R}_{++}^{S-1}$ by setting $p_1 \equiv 1$, and let the equilibrium correspondence $E : \mathbf{R}_{++}^{mS} \to \mathbf{R}_{++}^{mS} \times \mathbf{R}_{++}^{S-1}$ be defined by

$$E(\omega) = \{(x, p) \in \mathbf{R}_{++}^{mS} \times \mathbf{R}_{++}^{S-1} : (x, p) \text{ is an equilibrium in } \mathcal{E}(\omega)\}$$

Our basic notion of determinacy requires that for a given endowment profile ω, the economy has finitely many equilibria, and the equilibrium correspondence is continuous at ω.

Definition The economy $\mathcal{E}(\omega)$ is *determinate* if the number of equilibria is finite and the equilibrium correspondence $E : \mathbf{R}_{++}^{mS} \to \mathbf{R}_{++}^{mS} \times \mathbf{R}_{++}^{S-1}$ is continuous at ω.

The regularity conditions in Assumption D suffice to ensure that economies with variational preferences are generically determinate.

Theorem 1 *Under Assumptions A and D, there is a set $\Omega \subset \mathbf{R}_{++}^{mS}$ of full Lebesgue measure such that the economy $\mathcal{E}(\omega)$ is determinate for all $\omega \in \Omega$.*

Proof: We begin by characterizing properties of individual and excess demand under Assumptions A and D. To that end, first fix i and define the function $h^i : \mathbf{R}_{++}^S \to \mathbf{R}$ by

$$h^i(u) := \min_{\pi \in \Delta} \{\pi \cdot u + c^i(\pi)\}$$

Notice that h^i is concave, hence $Dh^i(u)$ and $D^2h^i(u)$ exist for almost all u, and $D^2h^i(u)$ is negative semidefinite where defined. In analogy with the notation above, let

$$M^i(u) := \{\pi \in \Delta : \pi \in \arg\min_{\pi \in \Delta} \{\pi \cdot u + c^i(\pi)\}\}.$$

By Clarke (1983, 2.8, Cor. 2), $\partial h^i(u) = M^i(u)$ for each $u \in \mathbf{R}_+^S$. In addition, the concavity of h^i means that $\partial h^i(u) = \{Dh^i(u)\}$ for almost every u, from which it follows that $M^i(u)$ is a singleton for almost every u.

Now by definition $V^i(x) \equiv h^i(U^i(x))$. As V^i is concave, $DV^i(x)$ exists for almost every $x \in \mathbf{R}_+^S$, and

$$DV^i(x) = Dh^i(U^i(x))DU^i(x)$$

for almost every x. In addition, $D^2V^i(x)$ exists almost everywhere, and

$$
\begin{aligned}
D^2V^i(x) &= Dh^i(U^i(x))D^2U^i(x) + D^2h^i(U^i(x))DU^i(x)DU^i(x) \\
&= Dh^i(U^i(x))D^2U^i(x) + DU^i(x)D^2h^i(U^i(x))DU^i(x)
\end{aligned}
$$

where the second equation follows from the fact that $DU^i(x)$ is a diagonal matrix. The concavity of h^i together with Assumption A ensures that the second term above is negative semidefinite. From Assumptions A, D, and the observation that $Dh^i(U^i(x)) = M^i(x)$, which is strictly positive by Lemma 2, the first term is negative definite. Thus, there exists a set D^i of full measure on which $D^2V^i(x)$ is defined and negative definite.

For any $x \in D^i$, the bordered Hessian

$$H(x) \equiv \begin{pmatrix} D^2V^i(x) & DV^i(x)^T \\ DV^i(x) & 0 \end{pmatrix}$$

is nonsingular. To see this, suppose $H(x)z = 0$ for some $z \in \mathbf{R}^{S+1}$. Write $z = (\tilde{z}, z_{S+1})$ where $\tilde{z} \in \mathbf{R}^S$, and note that $H(x)z = 0$ implies that

$$
\begin{aligned}
D^2V^i(x)\tilde{z} + z_{S+1}DV^i(x)^T &= 0 \\
DV^i(x) \cdot \tilde{z} &= 0
\end{aligned}
$$

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By standard arguments, these equations imply that

$$\langle \tilde{z}, D^2 V^i(x)\tilde{z}\rangle + z_{S+1}\left(DV^i(x)^T \cdot \tilde{z}\right) = \langle \tilde{z}, D^2 V^i(x)\tilde{z}\rangle = 0$$

Since $D^2 V^i(x)$ is negative definite, we conclude that $\tilde{z} = 0$. Then $z_{S+1} = 0$ as well. Hence $H(x)$ is nonsingular for any $x \in D^i$.

Let $x^i : \mathbf{R}^{S-1}_{++} \times \mathbf{R}_{++} \to \mathbf{R}^S_+$ denote the demand function of agent i. By Proposition 1 in Pascoa and Werlang (1999), for each i there exists a set $L^i \subset \mathbf{R}^{S-1}_+$ of Lebesgue measure zero such that x^i is approximately pointwise Lipschitz continuous on $A^i := (\mathbf{R}^{S-1}_{++} \times \mathbf{R}^{mS}_{++}) \setminus (x^i)^{-1}(L^i)$.[4] Set $L = \cup_i L^i$ and $A = \cap_i A^i$.

Following Debreu (1970) argument for generic determinacy in the case of C^1 excess demand, define $F : \mathbf{R}^{S-1}_{++} \times \mathbf{R}_{++} \times \mathbf{R}^{(m-1)S}_{++} \to \mathbf{R}^{mS}$ by

$$F(p, w_1, z_2, \ldots, z_m) = (x^1(p, w_1) + \sum_{i=2}^{m} x^i(p, p \cdot z_i) - \sum_{i=2}^{m} z_i, z_2, \ldots, z_m).$$

Note that given any initial endowment vector $\omega = (\omega^1, \ldots, \omega^m) \in \mathbf{R}^{mS}_{++}$, $F(p, w_1, z_2, \ldots, z_m) = \omega$ if and only if p is an equilibrium price vector for $\mathcal{E}(\omega)$, $z_i = \omega_i$ for $i = 2, \ldots, m$, and $w_1 = p \cdot \omega_1$. Moreover, note that for all $p \in \mathbf{R}^{S-1}_{++}$, for all $w_1 \in \mathbf{R}_{++}$ and for all $(z_2, \ldots, z_m) \in \mathbf{R}^{(m-1)S}_{++}$,

$$p \cdot [x^1(p, w_1) + \sum_{i=2}^{m} x^i(p, p \cdot z_i) - \sum_{i=2}^{m} z_i] = w_1.$$

Next, we use these observations to show that if ω is a regular value of F as defined in Shannon (1994), then the economy $\mathcal{E}(\omega)$ is determinate. To that end, suppose that ω is a regular value of F, so for each $y = (p, w_1, z_2, \ldots, z_m) \in F^{-1}(\omega)$, $DF(y)$ exists and has full rank. By Theorem 15 of Shannon (1994), $\mathcal{E}(\omega)$ has finitely many equilibria, and there exists a compact set $Y \subset \mathbf{R}^{S-1}_{++} \times \mathbf{R}_{++}$ such that $\deg(F, Y, \omega) = \sum_{y \in F^{-1}(\omega)} \operatorname{sgn} \det DF(y) = 1$. Using the assumption that ω is a regular value of F again, each solution $y \in F^{-1}(\omega)$ is essential in the sense of Fort (1950), from which the continuity of E at ω follows.

Now the result follows provided the set of regular values of F has full Lebesgue measure. To establish this, we adapt a portion of the proof of Proposition 2 in Pascoa and Werlang (1999). For each i, there exists a set $M^i \subset \mathbf{R}^S_+$ of full measure such that $M^i = \cup_{k=1}^{\infty} M^i_k$ where $\{M^i_k\}$ is a countable collection of disjoint sets such that for each k, $x^i|_{O^i_k}$ is differentiable where $O^i_k := (x^i)^{-1}(M^i_k)$. For each k, set $\tilde{M}^1_k = \{\omega \in \mathbf{R}^{mS}_+ : \omega = F(p, w_1, \omega_2, \ldots, \omega_m)$ and $x^1(p, w_1) \in M^1_k\}$ and for $i \neq 1$, set $\tilde{M}^i_k = \{\omega \in \mathbf{R}^{mS}_+ : \omega = F(p, w_1, \omega_2, \ldots, \omega_m)$ and

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[4]A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is *approximately pointwise Lipschitz continuous* on $A \subset \mathbf{R}^n$ if for each $x \in A$, ap $\limsup_{y \to x}(\|f(y) - f(x)\|)/(\|y - x\|) < \infty$, where for a function $g : \mathbf{R}^n \to \mathbf{R}$, the *approximate lim sup* of g at x is defined by ap $\limsup_{y \to x} g(y) := \inf\{t \in \mathbf{R} : \{y \in \mathbf{R}^n : g(y) > t\}$ has density zero at $x\}$. See Pascoa and Werlang (1999) and Federer (1996) for more details.

$x^i(p, p \cdot \omega_i) \in M_k^i$}. For each i, set $\tilde{M}^i = \cup_k \tilde{M}_k^i$ and set $\tilde{M} = \cap_{i=1}^m \tilde{M}^i$. Now it suffices to show that $\tilde{M}$ has full measure in the range of F.

To that end, let $\rho : \mathbf{R}_+^{mS} \to \mathbf{R}^{mS}$ be given by $\rho(z_1, \ldots, z_m) = (z_1 + \cdots + z_m, z_2, \ldots, z_m)$. Since ρ is linear and invertible, ρ maps sets of full measure into sets of full measure. Thus $\rho((\cap_i M^i)^m)$ has full measure in $\mathbf{R}_+^{mS}$. Consequently, $\rho_1((\cap_i M^i)^m)$ must have full measure in $\mathbf{R}_+^S$. Set $A := \rho_1((\cap_i M^i)^m) \times \mathbf{R}_+^{mS}$. Then $\tilde{M} = $ range F $\cap \Psi_1(A)$ where $\Psi : \mathbf{R}_+^{mS} \to \mathbf{R}^{mS}$ is given by $\Psi(y_1, \ldots, y_m) = (y_1 - \sum_{i=2}^m y_i, y_2, \ldots, y_m)$. As with ρ, Ψ maps sets of full measure into sets of full measure, from which it follows that $\Psi_1(A)$ has full measure. Thus range $F \setminus \tilde{M}$ has measure zero.

By Sard (1958), the set of critical values of F in $\tilde{M}$ has measure zero. As range $F \setminus \tilde{M}$ has measure zero, the proof is completed. ∎

Under stronger conditions on preferences, we can derive a somewhat stronger result regarding the generic properties of the equilibrium correspondence. We give one such result below. For this result, we consider the case in which each consumer's preference order is represented by a Choquet expected utility function with a convex capacity.

(DC) for each i, $c^i = c_{\Pi^i}$ for a set $\Pi^i \subset \Delta \cap \mathbf{R}_{++}^S$ that is the core of a convex capacity on S.

Under Assumptions D and DC, each consumer's utility is a Choquet expected utility function of the form
$$V^i(x) = \min_{\pi \in \Pi^i} E_\pi[u^i(x)]$$
where $\Pi^i \subset \Delta \cap \mathbf{R}_{++}^S$ is the core of a convex capacity. This additional assumption provides extra structure on individual demand, and hence on excess demand, which allows us to sharpen our generic determinacy result. For this we make use of the following stronger notion of determinacy.

Definition The economy $\mathcal{E}(\omega)$ is *Lipschitz determinate* if the number of equilibria is finite and the equilibrium correspondence $E : \mathbf{R}_{++}^{mS} \to \mathbf{R}_{++}^{mS} \times \mathbf{R}_{++}^{S-1}$ is locally Lipschitz continuous at ω, that is, for every equilibrium $(x, p) \in E(\omega)$ there exist neighborhoods O of (x, p) and W of ω such that every selection from $O \cap E$ is locally Lipschitz continuous on W.

Theorem 2 *Under Assumptions A, D, and DC, there is a set $\Omega \subset \mathbf{R}_{++}^{mS}$ of full Lebesgue measure such that the economy $\mathcal{E}(\omega)$ is Lipschitz determinate for all $\omega \in \Omega$.*

Proof: By Theorem 17 in Shannon (1994), it suffices to show that each demand function x^i is locally Lipschitz continuous. To see that this follows from Assumptions D and DC, let $\mathcal{O}$

denote the set of all ordered partitions of S.[5] Given $x \in \mathbf{R}_+^S$, define the ordered partition generated by x as $O(x) = \{A_1, \dots, A_k\}$ where for each $j = 1, \dots, k$:

(i) $x_s = x_t$ for all $s, t \in A_j$

(ii) $x_s < x_t$ for all s, t such that $s \in A_j$ and $t \in A_{j+1}$

Given $O \in \mathcal{O}$, define
$$X_O := \{x \in \mathbf{R}_{++}^S : O(x) = O\}.$$

Fix $O \in \mathcal{O}$. For each $p \in \mathbf{R}_{++}^{S-1}$, let

$$x_O^i(p) = \quad \arg\max_{x \in X_O} \left[\min_{\pi \in \Pi^i} E_\pi[u^i(x)] \right]$$
$$\text{s.t. } p \cdot x \le p \cdot \omega^i$$

Note that $\mathcal{O}$ is finite, and $\mathbf{R}_{++}^S = \cup_{O \in \mathcal{O}} X_O$. In addition, X_O is convex for each $O \in \mathcal{O}$, and isomorphic to an open subset of $\mathbf{R}_{++}^{|O|}$. For each $O \in \mathcal{O}$, there exists $M_O \subset \Pi^i$ such that for every $x, x' \in X_O$, $M_i(x) = M_i(x') = M_O$, and

$$V^i(x) = E_\pi[u^i(x)]$$

for any $\pi \in M_O$. Thus for each $O \in \mathcal{O}$, $x_O^i(\cdot)$ is C^1, and hence locally Lipschitz continuous. For each p, $x^i(p) \equiv x_{O(x^i(p))}^i(p)$, so $x^i(p) \in \cup_{O \in \mathcal{O}} x_O^i(p)$, i.e., $x^i(\cdot)$ is a selection from $\cup_{O \in \mathcal{O}} x_O^i(\cdot)$. Finally, $x^i(\cdot)$ is continuous, by standard arguments.

Putting these observations together, the result then follows immediately from the fact that $x^i(\cdot)$ is a continuous selection from $\cup_{O \in \mathcal{O}} x_O^i(\cdot)$, as $\mathcal{O}$ is finite and each $x_O^i(\cdot)$ is locally Lipschitz continuous (for example, see Mas-Colell (1985)). ∎

Remark 1: A similar result could be derived in a multiple priors model under other restrictions on the sets Π^i, such as having a smooth boundary with no critical points. Since no axiomatization yields a representation with these sorts of restrictions on the set of priors Π^i, we do not pursue such results further here. Alternatively, the methods of Blume and Zame (1993) could be used in the "plausible priors" model axiomatized by Siniscalchi (2006), which gives rise to a convex set of priors with finitely many extreme points, provided the utility for consumption is analytic and concave. This would provide an alternative proof of Theorem 1 for this particularly interesting subclass of multiple priors preferences.

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[5]An *ordered partition* is a partition in which the order of sets matters. Thus while $\{\{1, 2\}, \{3\}, \{4, 5\}\}$ and $\{\{3\}, \{1, 2\}, \{4, 5\}\}$ are identical partitions of the set $\{1, 2, 3, 4, 5\}$, they are distinct ordered partitions.

Remark 2: Other classes of preferences can be easily combined with these results by noting that only the properties of individual and aggregate excess demand matter for the determinacy results we derive. For example, it is straightforward to give conditions on the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005) under which individual demand is continuously differentiable.[6] Theorems 1 and 2 remain valid if we enlarge the admissible class of preferences to include these as well. We have focused on the class of variational preferences because it efficiently nests the main models that have been studied in applications, and contains significant instances of kinks. This class allows for the sort of behavior that has been attributed to ambiguity while making determinacy results not obvious.

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[6]A similar comment applies to the models of Ergin and Gul (2004) and Nau (2006).

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