What's on the Menu?
Deciding What is Available to the Group*

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June 2010

Abstract

We consider the problem of aggregating menus. Each agent submits a menu of options, which the social rule aggregates into a single menu. First, we study a general environment where menus are arbitrary subsets, and characterize a union rule which takes the union of individual menus. Second, we study a probabilistic environment where menus are closed convex subsets of lotteries over options, and characterize a rule which takes a convex combination over individual menus with fixed weights.

1 Introduction

The basic problem of deciding a common set of options for a number of agents arises frequently. Consider the following:

- Two friends want to meet at a restaurant for dinner. One is on a diet and hopes to avoid unhealthy options, while the other has just received a promotion and hopes to celebrate by indulging in a rich meal.
- A government designs a mandatory pension system, to be funded through implicit forced savings taxed on current income. Some residents prefer the flexibility for current

*The authors thank Federico Echenique, Eddie Dekel, Ivana Komunjer, Rolf Schneider, and seminar participants at Carlo Alberto and CORE for helpful comments. We are especially grateful to Todd Sarver for detailed comments and suggestions.

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consumption afforded with smaller rates of forced savings. Others prefer larger rates to moderate their temptation to overconsume in the present. A single common system must be incorporated into the tax code.

- Many laws are basic restrictions on behavior applied consistently to everyone. Citizens disagree on limits of abortion rights, the right to free speech, and so on, but a single set of restrictions must be reached.

- Nations often meet to ratify treaties to limit their behavior. For example, the Geneva Conventions dictate rules of engagement during war. Some countries prefer to restrict torture of prisoners and attacks on civilians, while others would like more liberal rules allowing these practices.

Kreps (1979) introduced an axiomatic theory of decision making for a single agent over menus, which is extended by Dekel, Lipman, and Rustichini (2001) and Gul and Pesendorfer (2001). These theories incorporate a particular agent's desire for larger menus to provide flexibility or for smaller menus to provide commitment. A common theme in the prior examples is a social tension between flexibility and commitment across many agents. We study rules for resolving this tension.

In particular, we focus attention to a simple setting where each agent proposes or submits a menu, and the social rule aggregates these submissions into a common menu. An alternative approach might aggregate entire preferences over menus. For example, a variant of the classical Harsanyi (1955) Aggregation Theorem, e.g. De Meyer and Mongin (1995), implies that a Paretian, independent, and monotone social aggregation of independent and monotone preferences over sets of lotteries can be represented with a convex combination of the subjective state spaces and measures which represent individuals' preferences. One difficulty with implementing this rule is obtaining each individual's preference over menus. The main substantive motivation for the current setting is its relative simplicity. Each agent submits a single menu, rather than a rank order of the entire power set of menus.

In this paper, we consider two structural environments. First is the general environment where no structure is imposed on the space of outcomes, which is analogous to Kreps (1979). Here, we characterize a rule which returns the union of the individually proposed menus. Second is a probabilistic environment where the outcome space is the space of lotteries over deterministic outcomes, which is analogous to Dekel, Lipman, and Rustichini (2001) and Gul and Pesendorfer (2001) and provides some useful linear structure. Here, we characterize a rule which returns a convex combination with fixed weights over the individually proposed menus.

Another interpretation of the environment is a resolution of competing opinions or conjectures. The general model can be viewed in terms of a principal asking various advisors about what they view as the possible states of the world in a setting with unforeseen contingencies, while the probabilistic model corresponds to asking advisors for their sets of beliefs over states in a setting with ambiguity regarding probabilities. A corollary contribution of the paper is to provide characterizations of different possibility and belief aggregators in these settings.

We note that the general model considered here is formally similar to some other aggregation problems. Under approval voting (Brams and Fishburn 1978, Weber 1995), voters submit ballots of acceptable candidates from each voter and outputs as winners the candidates which appear on the most ballots. Aside from the obvious difference in interpretation, the resulting representation is also distinct.[1] The union rule considered in this paper would violate the Consistency condition required to characterize approval voting across populations (Fishburn 1978), while approval voting would violate the Monotonicity condition used here.

Finally, there is a small literature on axiomatic group definition, where each agent submits a subset of agents and the aggregation rule outputs a group of agents (Kasher and Rubinstein 1997, Schmeidler 2003).[2] The set of outcomes is the power set of agents, so corresponds to a special case of our environment. Many of the axioms in this literature invoke this connection, e.g. each agent can decide on her own identity, and are not meaningful in our general environment. In this special environment, our proposed union rule for the finite case can be interpreted is as an extremely permissive aggregation (any agent can approve i as a member of the group), or dually as an extremely restrictive aggregation (interpreting the submitted sets complementarily, any agent can veto i as a member of the group).

2 General model

We begin by describing the general model. Let $N = \{1, \dots, n\}$ denote a finite set of agents and X denote an arbitrary set of alternatives. In general $\mathcal{X} \subseteq 2^X$ denotes some universe of admissible subsets or menus. In this section, we will take $\mathcal{X} = 2^X \setminus \{\emptyset\}$ to be the family of all nonempty subsets of X. A rule is a function $F : \mathcal{X}^N \to \mathcal{X}$ assigning a menu to every profile of menus. One such rule is the **union rule** which maps each profile to the union of its components, $F(A_1, \dots, A_n) = \bigcup_{i \in N} A_i$. This is a liberal rule in the following sense: if any agent desires the freedom to choose a particular option, the social menu will include

[1]Moreover, approval voting is generally characterized with conditions applied across populations, while we focus on a single population setting.

[2]We thank Eddie Dekel for mentioning this work to us.

that option. Implicitly, any agent has veto power over the exclusion of an option; in the deciding limits on behavior, consensus is required before a law or treaty is ratified which makes some option unavailable to the group. In the corollary interpretation of the model as the contingencies foreseen by different agents, the rule has the straightforward and sensible implication that the contingencies foreseen to the group are those which are foreseen by any individual member.

We now consider some restrictions on rules.

Axiom 1 (Unanimity)**.** For all $A \in \mathcal{X}, F(A, \ldots, A) = A$.

Unanimity is a minimal efficiency criterion. If every individual agrees on the same menu, then that menu should be the social outcome.

Axiom 2 (Anonymity)**.** For any bijection $\tau : N \to N$ and profile $(A_1, \ldots, A_n) \in \mathcal{X}^N$, $F(A_1, \ldots, A_n) = F\left(A_{\tau(1)}, \ldots, A_{\tau(n)}\right)$.

This classical condition, which makes the identities of the agents irrelevant, captures some notion of equity. As the proofs will make clear, it is also very technically powerful in this setting.

Axiom 3 (Monotonicity)**.** If $A_i \subseteq B_i$ for all $i \in N$, then $F(A) \subseteq F(B)$.

Monotonicity asserts that if every agent submits a larger menu with more flexibility (or a smaller one with more commitment), then the social outcome should also allow for more flexibility (or impose more commitment).

Axiom 4 (Disjoint Additivity)**.** If $A_i \cap B_i = \emptyset$ for all $i \in N$, then $F(A_i \cup B_i, \ldots, A_n \cup B_n) = F(A_1, \ldots, B_n) \cup F(B_1, \ldots, B_n)$.

We can also invoke the following stronger but still necessary condition, applied at the level of the individual submission rather than the entire profile: if $A_i \cap B_i = \emptyset$ for some $i \in N$, then

$$F(A_i \cup B_i, (C_j)_{j \neq i}) = F(A_i, (C_j)_{j \neq i}) \cup F(B_i, (C_j)_{j \neq i}).$$

Thusly stated, Disjoint Additivity resembles a path independence or separability condition. The social menu induced by a profile of individual menus can be decomposed and analyzed through disjoint individual submenus. One way to understand this condition is to suppose we split the domain $X = X' \cup X''$, with X' and X'' disjoint, then ask some agent i to submit a menu $A_i \subseteq X'$ and $B_i \subseteq X''$ from each subdomain. For example, these menus might respectively be the beef and chicken dishes she would like to have available at a restaurant. If she selects a conditional menu $A_i \subseteq X'$ of beef dishes and a conditional menu $B_i \subseteq X''$ of chicken dishes, then the social menu that would have been offered if she had submitted the union $A_i \cup B_i$ of beef and chicken dishes is the combination of social menus that are offered based on her conditional menus, fixing all other agents' submissions. Of course, someone who submits A_i from X' and B_i from X'' might submit a separate menu not equal to $A_i \cup B_i$ from the grand domain X: for example, she finds some option $a \in A_i$ tempting against B_i in a way that it is not against other choices in A_i. The axiom asserts only that if she were to submit the union $A_i \cup B_i$ from the grand domain, then the resulting social menu would take the union of the conditional social menus.

As with path independence, any rule satisfying Disjoint Additivity enjoys a form of informational parsimony, since the aggregation can be separated into smaller domains. However, this separability, in conjunction with the three prior axioms, is satisfied only by the union rule.

Theorem 1. *Suppose $|X| \geq |N|$. A rule $F : \mathcal{X}^N \to \mathcal{X}$ satisfies Unanimity, Anonymity, Monotonicity, and Disjoint Additivity if and only if $F(A_1, \ldots, A_n) = \bigcup_{i \in N} A_i$.*

Proof. We omit the straightforward verification of necessity, moving directly to sufficiency.

We first prove the claim for profiles of singletons. Let $x_i \in X$ for each $i \in N$ and $A = \bigcup_{i \in N}\{x_i\}$. Let $m = |A|$ denote the number of distinct elements in the profile $(\{x_1\}, \ldots, \{x_n\})$. We need to show $F(\{x_1\}, \ldots, \{x_n\}) = A$. Since there are more objects than agents, $|X| \geq |N|$, there exists some residual set $B = \{b_1, \ldots, b_{n-m}\} \subseteq X \setminus \{x_1, \ldots, x_m\}$ such that $|B| = n - m$.

One set containment, $F(\{x_1\}, \ldots, \{x_n\}) \subseteq F(A, \ldots, A) \subseteq A$, follows from Monotonicity and Unanimity. This also implies that $F(\{x_1\}, \ldots, \{x_n\})$ is disjoint from B. So, to prove that $A \subseteq F(\{x\}, \ldots, \{x_n\})$, it suffices to show that $A \cup B \subseteq F(\{x_1\}, \ldots, \{x_n\}) \cup B$.

To this end, we may assume, by reordering and appealing to Anonymity, that the first m components of the profile are distinct: $x_i \neq x_j$ for all $i, j \leq m$. Consider the profile

$$(\{y_1\}, \ldots, \{y_n\}) = (\{x_1\}, \{x_2\}, \ldots, \{x_m\}, \{b_1\}, \{b_2\}, \ldots, \{b_{n-m}\}).$$

By construction, $y_i \neq y_j$ whenever $i \neq j$ and $\{y_1, \ldots, y_n\} = A \cup B$. Let τ^i denote the i-th cyclic permutation on N.[3]

$$
\begin{aligned}
A \cup B &= F(A \cup B, \ldots, A \cup B), && \text{by Unanimity} \\
&= F\left(\bigcup_{i \in N} \left\{y_{\tau^i(1)}\right\}, \ldots, \bigcup_{i \in N} \left\{y_{\tau^i(n)}\right\}\right) \\
&= \bigcup_{i \in N} F\left(\{y_{\tau^i(1)}\}, \ldots, \{y_{\tau^i(n)}\}\right), && \text{by } n \text{ applications of Disjoint Additivity} \\
&= F(\{y_1\}, \ldots, \{y_n\}), && \text{by Anonymity} \\
&\subseteq F(\{x_1\} \cup B, \ldots, \{x_n\} \cup B), && \text{by Monotonicity} \\
&= F(\{x_1\}, \ldots, \{x_n\}) \cup F(B, \ldots, B), && \text{by Disjoint Additivity} \\
&= F(\{x_1\}, \ldots, \{x_n\}) \cup B, && \text{by Unanimity.}
\end{aligned}
$$

Moving to the general case, take an arbitrary profile $(A_1, \ldots, A_n)$. By Monotonicity and Unanimity, $F(A_1, \ldots, A_n) \subseteq F\left(\bigcup_{i \in N} A_i, \ldots, \bigcup_{i \in N} A_i\right) = \bigcup_{i \in N} A_i$. For the other direction, suppose $x \in \bigcup_{i \in N} A_i$. Then $x \in A_i$ for some A_i, so there exists a selection $(x_1, \ldots, x_n) \in \prod_{i \in N} A_i$ such that $x \in \{x_1, \ldots, x_n\}$. But $x \in \{x_1, \ldots, x_n\} = F(\{x_1\}, \ldots, \{x_n\}) \subseteq F(A_1, \ldots, A_n)$, by the previous step and Monotonicity. So, we have $\bigcup_{i \in N} A_i \subseteq F(A_1, \ldots, A_n)$. $\square$

The simple verification of the logical independence of the axioms is omitted.

The hypothesis that the number of objects is at least as large as the number of agents is indispensable, as demonstrated by the following example.

Example 1. Let $X = \{x, y\}$ and $N = \{1, 2, 3\}$. If there exists some $i \in N$ such that $A_i = \{x, y\}$, let $F(A_1, A_2, A_3) = \{x, y\}$. There are four remaining profiles to consider, up to permutation of agents:

$$
\begin{aligned}
F(\{x\}, \{x\}, \{x\}) &= \{x\} \\
F(\{x\}, \{x\}, \{y\}) &= \{x\} \\
F(\{x\}, \{y\}, \{y\}) &= \{y\} \\
F(\{y\}, \{y\}, \{y\}) &= \{y\}.
\end{aligned}
$$

This example satisfies Unanimity, Anonymity, Monotonicity, and Disjoint Additivity.

The theorem can also be proved with a stronger Additivity condition whose hypothesis does not require that announced menus be disjoint, in which case the cardinality hypoth-

[3] So $\tau^i(k) = [k + i] \bmod n$.

esis is no longer required. This strengthened axiom implies both Disjoint Additivity and Monotonicity.

3 Probabilistic model

In this section, suppose X is a finite set. Let ΔX denote the set of lotteries over X. We now take $\mathcal{X} = \mathcal{K}(\Delta X)$, the nonempty, closed, and convex subsets of ΔX.[4] Define the convex combination of two sets $A, B \in \mathcal{K}$ with weight $\alpha \in [0, 1]$ by

$$\alpha A + (1 - \alpha)B = \{\alpha p + (1 - \alpha)q : p \in A, q \in B\}.$$

The additional linear structure here affords new kinds of aggregation rules. For example, the **weighting rule** takes a convex combination, with fixed weights across agents, of the individual menus: $F(A_1, \dots, A_n) = \sum_{i \in N} \mu_i A_i$, for some $\mu \in \Delta N$.

The linearized domain and the accompanying weighting rule exploit objective randomization. This randomization is similar to how a random allocation rule is used to improve ex ante equity in the assignment of indivisible objects: "Using a lottery is one of the oldest tricks (going further back than the Bible . . .) to restore fairness in such problems" (Bogomolnaia and Moulin 2001, p. 295). Moreover, in some settings this provides an accurate description of the options: financial assets can be summarized as induced distributions over returns; restaurants with changing menus or stores with uncertain inventories can be summarized as probabilities of different entrees or items being available. In other settings, the linearization can be interpreted as proportions of options, without appeal to randomization: the mutual funds available in a retirement plan can be considered as mixtures of specific assets.

This linearization also provides useful theoretical leverage. Beside the standard point regarding ex ante equity, randomization has a particular appeal in the context of flexibility and self-control in maintaining resoluteness. If each agent submits a single point, the social outcome arguably should reflect this strong desire for commitment. However, it is unclear how to sensibly select a particular singleton in a setting without randomization. In fact, Theorem 1 highlights a potential tension between separability and resoluteness. With randomization, the suggested weighting rules resolutely map any profile of singletons to a singleton; to do so, it requires an alternate form of separability, to be defined shortly, which involves the linearization of the domain.

This domain also has special appeal when interpreting the model as aggregating opinions

[4] We could also allow for sets which are not convex by assuming that $F(A_1, \dots, A_n) = F(\text{conv}(A_1), \dots, \text{conv}(A_n)$, where $\text{conv}(A_i)$ denotes the convex hull of A_i.

regarding uncertainty. For example, in the maxmin expected utility model, uncertainty is captured as a set of prior beliefs. If consulting different experts with subjective opinions regarding this set, their advice can be summarized as sets of probabilities.

The weighting rule fails the Disjoint Additivity condition of the union rule, but satisfies the following condition, which invokes the linear structure of the space of lotteries.

Axiom 5 (Mixture Linearity)**.** For all $(A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathcal{X}^N$ and $\alpha \in [0, 1]$,

$$F(\alpha A_1 + (1-\alpha)B_1, \ldots, \alpha A_n + (1-\alpha)B_n) = \alpha F(A_1, \ldots, A_n) + (1-\alpha)F(B_1, \ldots, B_n).$$

This axiom is an analog of independence for individual choice over menus of lotteries, as used Dekel, Lipman, and Rustichini (2001), applied to our social choice setting. Its normative motivation is also analogous, in terms of the timing of resolution of uncertainty. In this interpretation, the combination $(\alpha A_1 + (1-\alpha)B_1, \ldots, \alpha A_n + (1-\alpha)B_n)$ is viewed as a randomization, say generated by an α-weighted coin, between the profiles $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$. Then the social menu over these randomized submissions is the randomization of the social menu induced by the submissions $(A_1, \ldots A_n)$ and $(B_1, \ldots, B_n)$. So, a Mixture Linear rule is invariant to whether the coin is flipped before or after the point of aggregation. If indifference to the timing of the resolution of uncertainty is taken as normatively compelling, this assumption simply invokes this indifference in the aggregate.

The justification for Mixture Linearity invokes a form of independence over future options. Some recent work, e.g. Epstein, Marinacci, and Seo (2005), relaxes this assumption regarding preferences over menus. Nonetheless, the descriptive concerns which motivate these generalizations seem orthogonal to the aggregation problem of this paper, especially considering the normative motivation for Mixture Linearity here.

Replacing Disjoint Additivity with Mixture Linearity, we characterize the weighting rule with a uniform weighting over agents.

Theorem 2. *A rule $F : \mathcal{X}^N \to \mathcal{X}$ satisfies Unanimity, Anonymity, and Mixture Linearity if and only if $F(A_1, \ldots, A_n) = \sum_{i \in N} \frac{1}{n} A_i$.*

Proof. Let $(A_1, \ldots, A_n) \in \mathcal{X}^N$. Recall $\tau^i : N \to N$ denoted the i-th cyclic permutation on

N. Then:

$$\begin{aligned}
F(A_1, \dots, A_n) &= F(A_{\tau^i(1)}, \dots, A_{\tau^i(n)}), && \text{by Anonymity} \\
&= \sum_{i \in N} \frac{1}{n} F(A_{\tau^i(1)}, \dots, A_{\tau^i(n)}) \\
&= F\left(\frac{1}{n}\sum_{i \in N} A_{\tau^i(1)}, \dots, \frac{1}{n}\sum_{i \in N} A_{\tau^i(n)}\right), && \text{by Mixture Linearity} \\
&= F\left(\frac{1}{n}\sum_{i \in N} A_i, \dots, \frac{1}{n}\sum_{i \in N} A_i\right) \\
&= \frac{1}{n}\sum_{i \in N} A_i, && \text{by Unanimity.} \quad \square
\end{aligned}$$

Again, it is simple to verify that the three axioms are independent. The characterization makes no use of the Monotonicity condition which concerns commitment versus flexibility, and as such requires no appeal to interpretations requiring Monotonicity.

If the Anonymity axiom is dispensed, the weighting becomes arbitrary over agents. This is not a simple corollary of the Mixture Space Theorem or the standard Harsanyi Aggregation Theorem. In particular, the space of convex sets is somewhat intractable, and the proof moves the analysis to the space of support functions, following Dekel, Lipman, and Rustichini (2001). The consequent result can be interpreted as a version of the Aggregation Theorem over the infinite-dimensional space of sets.

Theorem 3. *A rule $F : \mathcal{X}^N \to \mathcal{X}$ satisfies Unanimity, Monotonicity, and Mixture Linearity if and only if there exists $\mu \in \Delta N$ such that $F(A_1, \dots, A_n) = \sum_{i \in N} \mu_i A_i$.*

Proof. We omit the straightforward verification of necessity and proceed to show sufficiency.

Let $m = |X|$ and $D = \{x \in \mathbb{R}^m : \|x\| = 1 \text{ and } \sum_i x_i = 0\}$. For any $A \in \mathcal{X}$, define the support function $\sigma_A : D \to \mathbb{R}$ of a set $A \in \mathcal{X}$ by $\sigma_A(s) = \max_{a \in A} a \cdot s$. Let $\mathcal{X}^* = \{\sigma_A : A \in \mathcal{X}\} \subseteq \mathbb{R}^D$ denote the family of all support functions induced by some set in $\mathcal{X}$. Define the dual function $F^* : [\mathcal{X}^*]^N \to \mathcal{X}^*$ by

$$F^*(\sigma_{A_1}, \dots, \sigma_{A_n}) = \sigma_{F(A_1, \dots, A_n)}.$$

It suffices to prove that $F^*(\sigma_{A_1}, \dots, \sigma_{A_n}) = \sum_{i \in N} \mu_i \sigma_{A_i}$.

The dual function inherits mixture linearity:

$$
\begin{aligned}
&F^{*}\left(\alpha\sigma_{A_1} + (1-\alpha)\,\sigma_{B_1}, \ldots, \alpha\sigma_{A_n} + (1-\alpha)\,\sigma_{B_n}\right) \\
&= F^{*}\left(\sigma_{\alpha A_1+(1-\alpha)B_1}, \ldots, \sigma_{\alpha A_n+(1-\alpha)B_n}\right) \\
&= \sigma_{F(\alpha A_1+(1-\alpha)B_1,\ldots,\alpha A_n+(1-\alpha)B_n)} \\
&= \sigma_{\alpha F(A_1,\ldots,A_n)+(1-\alpha)F(B_1,\ldots,B_n)} \\
&= \alpha\sigma_{F(A_1,\ldots,A_n)} + (1-\alpha)\sigma_{F(B_1,\ldots,B_n)} \\
&= \alpha F^{*}\left(\sigma_{A_1}, \ldots, \sigma_{A_n}\right) + (1-\alpha)\,F^{*}\left(\sigma_{B_1}, \ldots, \sigma_{B_n}\right).
\end{aligned}
$$

The support function also carries a form of monotonicity to the dual, since $\sigma_A \leq \sigma_{A'}$ if and only if $A \subseteq A'$. So, if $\sigma_{A_i} \leq \sigma_{B_i}$, i.e. $A_i \subseteq B_i$, for all $i \in N$, then:

$$F^{*}(\sigma_{A_1}, \ldots, \sigma_{A_n}) = \sigma_{F(A_1,\ldots,A_n)} \leq \sigma_{F(B_1,\ldots,B_n)} = F^{*}(\sigma_{B_1}, \ldots, \sigma_{B_n}).$$

Now, for any $s \in D$, consider the real-valued function $F^{*}_{s} : [\mathcal{X}^{*}]^{N} \to \mathbb{R}$ defined by

$$F^{*}_{s}(\sigma_{A_1}, \ldots, \sigma_{A_n}) = [F^{*}(\sigma_{A_1}, \ldots, \sigma_{A_n})](s).$$

This function is also mixture linear and monotonic. Note that

$$F^{*}_{s}(0, \ldots, 0) = 0.$$

To see this, let $\ell^{0} = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$ denote the uniform lottery over X and observe:

$$F^{*}(0, \ldots, 0) = F^{*}\left(\sigma_{\{\ell_0\}}, \ldots, \sigma_{\{\ell_0\}}\right) = \sigma_{\{\ell_0\}} = 0.$$

The argument for Theorems 1 and 2 of Dekel, Lipman, Rustichini, and Sarver (2007) can be replicated, with some obvious adjustments to account for the product space of the domain, to produce an increasing continuous linear extension $W_s : [C(D)]^{N} \to \mathbb{R}$ of F^{*}_{s} to $[C(D)]^{N}$, the space of all profiles of continuous real-valued functions on D. Therefore, by the Riesz Representation Theorem, there exists a profile $(\mu^{s}_{1}, \ldots, \mu^{s}_{n})$ of finite nonnegative measures on D such that

$$F^{*}_{s}(\sigma_{A_1}, \ldots, \sigma_{A_n}) = W_s(\sigma_{A_1}, \ldots, \sigma_{A_n}) = \sum_{i \in N} \int_{D} \sigma_{A_i}(t)\, d\mu^{s}_{i}(t).$$

Moreover, the measure μ^{s}_{i} is concentrated on $\{s\}$. To see this, recall that ℓ^{0} is the uniform distribution and consider the closed convex set $A = \{\ell \in \Delta X : \ell \cdot s \leq 0\}$. Let $t \in D \setminus \{s\}$.

Note that there exists some $\ell \in A$ for which $\ell \cdot t > 0$, so that $\sigma_A(t) > 0$.

Then, by Unanimity:

$$\begin{aligned}
\sigma_A(s) &= \sigma_{\ell^0}(s) \\
F^*_s(\sigma_A, \ldots, \sigma_A) &= F^*_s(\sigma_{\{\ell^0\}}, \ldots, \sigma_{\{\ell^0\}}) \\
\sum_{i \in N} \int_D \sigma_A(t)\, d\mu^s_i(t) &= \sum_{i \in N} \int_D \sigma_{\{\ell^0\}}(t)\, d\mu^s_i(t) \\
\sum_{i \in N} \left[\sigma_A(t)\mu^s_i(\{s\}) + \int_{D\setminus\{s\}} \sigma_A(s)\, d\mu^s_i(t)\right] &= \sum_{i \in N} \left[\sigma_{\{\ell^0\}}(s)\mu^s_i(\{s\}) + \int_{D\setminus\{s\}} \sigma_{\{\ell^0\}}(t)\, d\mu^s_i(t)\right] \\
\sum_{i \in N} \int_{D\setminus\{s\}} \sigma_A(t)\, d\mu^s_i(t) &= \sum_{i \in N} \int_{D\setminus\{s\}} \sigma_{\{\ell^0\}}(t)\, d\mu^s_i(t).
\end{aligned}$$

But, since $\sigma_A > \sigma_{\{\ell^0\}}$ on $D \setminus \{s\}$ and every μ^s_i is nonnegative, this suffices to show $\mu^s_i(D \setminus \{s\}) = 0$, as the right hand side of the final equality is zero. So, let $\mu^s_i = \mu^s_i(\{s\})$ and note that Unanimity implies $\sum_{i \in N} \mu^s_i = 1$.

We lastly show that $\mu^s = \mu^t$ for all $s, t \in D$. As a first step, observe that if $(\{x_1\}, \ldots, \{x_n\})$ is a profile of singletons, then $F(\{x_1\}, \ldots, \{x_n\})$ is a singleton. To see this, note that by Mixture Linearity and Unanimity,

$$\sum_{i \in N} \frac{1}{n} F(\{x_{\sigma^i(1)}\}, \ldots, \{x_{\sigma^i(n)}\}) = \left\{ \sum_{i \in N} \frac{1}{n} x_{\sigma^i(n)} \right\}.$$

Since the right hand side is a singleton, each component of the sum on the left hand side must also be a singleton.

To establish that $\mu^s = \mu^t$ for all s, t, we will consider two cases, the case in which $s = -t$ and the case in which $s \neq -t$. We will show that for any $i \in N$, $\mu^i_s = \mu^i_t$.

Recall that ℓ^0 is the uniform distribution over X. Let $i \in N$. Let $(A_1, \ldots, A_n)$ be a profile of singletons such that for all $j \neq i$, $A_j = \{\ell^0\}$, and A_i is any singleton. We claim that there exists some $\mu^i \in [0,1]$ such that $F(A_1, \ldots, A_n) = \mu_i A_i + (1 - \mu_i)\{\ell^0\}$. But this follows trivially by Unanimity, Monotonicity, and the prior observation that singleton profiles map to singletons.

Now, suppose that $s = -t$. Let us choose ε small enough so that $\ell^0 + \varepsilon s \in \Delta X$. Consider the profile where $A_i = \{\ell^0 + \varepsilon s\}$ and for all $j \neq i$, $A_j = \{\ell^0\}$. Then $F(A_1, \ldots, A_n) = \mu_i\{\ell^0 + \varepsilon s\} + (1 - \mu_i)\{\ell^0\}$ for some μ_i. In particular, as $\sigma_{\ell^0}(s') = 0$, we may conclude that $F^*_s(\sigma_{A_1}, \ldots, \sigma_{A_n}) = \mu_i \varepsilon$ and $F^*_t(\sigma_{A_1}, \ldots, \sigma_{A_n}) = -\mu_i \varepsilon$. Consequently, $F^*_s(\sigma_{A_1}, \ldots, \sigma_{A_n}) =$

$-F^*_t(\sigma_{A_1}, \ldots, \sigma_{A_n})$. Now,

$$\mu_i^s = \frac{\mu_i^s \varepsilon}{\varepsilon} = \frac{\mu_i^s \sigma_{A_i}(s) + \sum_{j\neq i} \mu_j^s \sigma_{A_j}(s)}{\varepsilon} = \frac{F^*_s(\sigma_{A_1}, \ldots, \sigma_{A_n})}{\varepsilon}$$

and

$$\mu_i^t = \frac{\mu_i^t(-\varepsilon)}{-\varepsilon} = \frac{\mu_i^t \sigma_{A_i}(t) + \sum_{j\neq i} \mu_j^t \sigma_{A_j}(t)}{-\varepsilon} = \frac{F^*_t(\sigma_{A_1}, \ldots, \sigma_{A_n})}{-\varepsilon},$$

from which we conclude $\mu_i^t = \mu_i^s$.

Next consider the case where $s \neq -t$. For sufficiently small $\varepsilon > 0$, $\ell^0 + \varepsilon(s+t) \in \Delta X$. Then:

$$\sigma_{\{\ell^0+\varepsilon(s+t)\}}(s) = \sigma_{\{\ell^0\}}(s) + \sigma_{\{\varepsilon(s+t)\}}(s) = \varepsilon(1 + t \cdot s) = \sigma_{\{\ell+\varepsilon(s+t)\}}(t)$$

Consider the profile $(A_1, \ldots, A_n)$ where $A_i = \{\ell^0 + \varepsilon(s+t)\}$ and $A_j = \{\ell^0\}$ for all $j \neq i$. As previously, let us write $F(A_1, \ldots, A_n) = \mu_i\{\ell^0 + \varepsilon(s+t)\} + (1 - \mu_i)\{\ell^0\}$. Therefore, $F^*_s(\sigma_{A_1}, \ldots, \sigma_{A_n}) = \mu_i \varepsilon(1 + s \cdot t) = F^*_t(\sigma_{A_1}, \ldots, \sigma_{A_n})$.

Then:

$$\mu_i^s = \frac{\mu_i^s \varepsilon(1+t\cdot s)}{\varepsilon(1+t\cdot s)} = \frac{\mu_i^s \sigma_{A_i}(s) + \sum_{j\neq i} \mu_j^s \sigma_{A_j}(s)}{\varepsilon(1+t\cdot s)} = \frac{F^*_s(\sigma_{A_1}, \ldots, \sigma_{A_n})}{\varepsilon(1+t\cdot s)}$$

and

$$\mu_i^t = \frac{\mu_i^t \varepsilon(1+t\cdot s)}{\varepsilon(1+t\cdot s)} = \frac{\mu_i^t \sigma_{A_i}(t) + \sum_{j\neq i} \mu_j^t \sigma_{A_j}(t)}{\varepsilon(1+t\cdot s)} = \frac{F^*_t(\sigma_{A_1}, \ldots, \sigma_{A_n})}{\varepsilon(1+t\cdot s)}$$

As $F^*_s(\sigma_{A_1}, \ldots, \sigma_{A_n}) = F^*_t(\sigma_{A_1}, \ldots, \sigma_{A_n})$, we may conclude $\mu_i^s = \mu_i^t$.

Hence, there exists a $\mu \in \Delta N$ such that $[F^*(\sigma_{A_1}, \ldots, \sigma_{A_n})](s) = \sum_{i\in N} \mu_i \sigma_{A_i}(s)$ for all $s \in D$, as desired. $\square$

Finally, we note a question our analysis leaves open, in the hope that it is resolved by future research. Monotonicity is crucial in the proof to ensure that the linear extension of the dual function is continuous, hence submits to the Riesz Representation Theorem. We are unsure whether Monotonicity is redundant: we could neither prove the result without it nor construct an example which satisfies the other conditions but fails Monotonicity. The subtlety of the problem will be familiar to decision theorists working with similar mathematical tools. While Monotonicity seems normatively innocuous, it would be technically satisfying to resolve this question conclusively.

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