Problem 5.

Let's start with the second expression for $b$ and prove that it is equivalent to the third, and then that the third is equivalent to the first:

\[
b = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i [(X_i - \bar{X})Y_i - (X_i - \bar{X})\bar{Y}]}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i [(X_i - \bar{X})Y_i] - \sum_i (X_i - \bar{X})\bar{Y}}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i [(X_i - \bar{X})Y_i] - \sum_i X_i \bar{Y} + \sum_i \bar{X}\bar{Y}}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i [(X_i - \bar{X})Y_i] - \bar{Y} \sum_i X_i + n \bar{X}\bar{Y}}{\sum_i (X_i - \bar{X})^2}
\]

At this point, we can use the fact that $\bar{X} = \frac{1}{n} \sum_i X_i$, and thus that $\sum_i X_i = n \bar{X}$. Hence,

\[
b = \frac{\sum_i [(X_i - \bar{X})Y_i] - \bar{Y}(n \bar{X}) + n \bar{X}\bar{Y}}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i [(X_i - \bar{X})Y_i]}{\sum_i (X_i - \bar{X})^2}
\]

as desired.

Now to prove that this expression is equivalent to the first:

\[
b = \frac{\sum_i [(X_i - \bar{X})Y_i]}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i [X_iY_i - \bar{X}Y_i]}{\sum_i [X_i^2 - 2\bar{X}X_i + \bar{X}^2]} = \frac{\sum_i X_iY_i - \bar{X} \sum_i Y_i}{\sum_i X_i^2 - \sum_i 2\bar{X}X_i + \sum_i \bar{X}^2} = \frac{\sum_i X_iY_i - \bar{X} \sum_i Y_i}{\sum_i X_i^2 - 2\bar{X} \sum_i X_i + n \bar{X}^2}
\]

Here again, use the fact that $\sum_i X_i = n \bar{X}$ and, similarly, that $\sum_i Y_i = n \bar{Y}$:
\[ b = \frac{\sum_i X_i Y_i - \bar{X}(n\bar{Y})}{\sum_i X_i^2 - 2\bar{X}(n\bar{X}) + n\bar{X}^2} \\
= \frac{\sum_i X_i Y_i - n\bar{X}\bar{Y}}{\sum_i X_i^2 - 2n\bar{X}^2 + n\bar{X}^2} \\
= \frac{\sum_i X_i Y_i - n\bar{X}\bar{Y}}{\sum_i X_i^2 - n\bar{X}^2} \]

as required.