Topics

- Formalities.
- Reduced strategic form.
- Backward induction and subgame perfection.
The need for refinements of Nash equilibrium

The concept of $NE$ is unsatisfactory since it

- ignores the sequential structure of the decision problems, and
- in sequential decision problems not all $NE$ are self-enforcing.

The following refinements have been proposed:

- subgame perfect, perfect, sequential, perfect sequential, proper
- persistent, justifiable, neologism proof, stable, intuitive, divine, undefeated and explicable.

All the refinements represent attempts to formulize the same two or three intuitive ideas (Kohlberg 1990).
Formalities (O 5.1-5.2, OR 6.1)

Definition

An extensive game with perfect information $\Gamma = \langle N, H, P, (\succeq_i) \rangle$ consists of

- A set $N$ of players.
- A finite or infinite set $H$ of sequences (histories), each component an action taken by a player.
- A player function $P : H \setminus Z \to N$ s.t. $P(h)$ being the player who takes an action after history $h$.
- A preference relation $\succeq_i$ on $Z$ for each player $i \in N$ where,
  
  The empty sequence $\emptyset$ is a member of $H$.
  
  If $(a^k)^L_{k=1} \in H$ then $(a^k)^L_{k=1} \in H$ for any $L < K$.
  
  If $(a^k)^\infty_{k=1}$ satisfies $(a^k)^L_{k=1} \in H$ for any $L$ then $(a^k)^\infty_{k=1} \in H$.

And,

- A set of terminal histories $Z \subseteq H$ s.t. $(a^k)^K_{k=1} \in Z$ if it is infinite, or
  $\not\exists a^{K+1}$ s.t. $(a^k)^{K+1}_{k=1} \in H$.
- If $h$ is a history of length $k$ then $(h, a)$ is a history of length $k + 1$
  consists of $h$ followed by $a$.

If the longest history is finite then the game has a finite horizon.
Strategies and outcomes

A strategy $s_i$ of player $i$ is a plan that specifies the action taken for every $h \in H \setminus Z$ for which $P(h) = i$.

For any $s = (s_i)_{i \in N}$, the outcome $O(s)$ of $s$ is $h \in Z$ that results when each player $i \in N$ follows $s_i$.

Nash equilibrium (O 5.3)

A $NE$ of $\Gamma = \langle N, H, P, (\succ_i) \rangle$ is a strategy profile $s^*$ s.t. for any $i \in N$

$$O(s^*) \succ_i O(s_i, s^*_{-i}) \forall s_i$$

Note that

- strategies are once-in-a-lifetime decisions made before the game starts.
- non-self-enforcing outcome (Selten 96.2).
The (reduced) strategic form

Consider an extensive game \( \Gamma = \langle N, H, P, (\succ_i) \rangle \)

The strategic form of \( \Gamma \) is a game \( \langle N, (S_i), (\succ'_i) \rangle \) in which for each \( i \in N \)

- \( S_i \) is player \( i \)'s strategy set in \( \Gamma \).
- \( \succ'_i \) is defined by

\[
s \succ'_i s' \iff O(s) \succ'_i O(s') \forall s, s' \in \times_{i \in N} S_i
\]

The reduced strategic form of \( \Gamma \) is a game \( \langle N, (S'_i), (\succ''_i) \rangle \) in which for each \( i \in N \)

- \( S'_i \) contains one member of equivalent strategies in \( S_i \), i.e., \( s_i \in S_i \)
  and \( s'_i \in S_i \) are equivalent if

\[
(s_{-i}, s_i) \sim'_j (s_{-i}, s'_i) \forall j \in N
\]

- \( \succ''_i \) defined over \( \times_{j \in N} S'_j \) and induced by \( \succ'_i \).
Subgame perfection (O 5.4 OR 6.2)


A subgame of $\Gamma$ that follows the history $h$ is the game $\Gamma(h)$

$$\langle N, H|_h, P|_h, (\succeq_i|_h) \rangle$$

where for each $h' \in H|_h$

$$(h, h') \in H, P|_h(h') = P(h, h')$$

and

$$h' \succeq_i|_h h'' \iff (h, h') \succeq_i (h, h'')$$

$s^*$ is a subgame perfect equilibrium (SPE) of $\Gamma$ if

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succeq_i|_h O_h(s_i|_h, s_{-i}|_h)$$

for each $i \in N$ and $h \in H \setminus Z$ for which $P(h) = i$ and for any $s_i|_h$ .

The equilibrium of the full game must induce on equilibrium on every subgame.
Backward induction
An algorithm for calculating the set of $SPE$ (Zermelo 1912)

- make payoff-maximizing choices at nodes which are one move from the end
- given those, make payoff-maximizing choices at nodes which are two move from the end,
- and so on.

$SPE$ eliminates $NE$ in which players’ threats are not credible (non-self-enforcing).

Kuhn’s theorems
Consider a finite extensive game with perfect information $\Gamma$

(Kuhn’s theorem) $\Gamma$ has a $SPE$.

- The proof is by backwards induction.
- Kuhn makes no claim about uniqueness.

$\Gamma$ has a unique $SPE$ if there is no $i \in N$ and $z, z' \in Z$ such that $z \sim_i z'$.

$\Gamma$ is dominance solvable if

$$z \sim_i z' \exists i \in N \Rightarrow z \sim_j z' \forall j \in N$$

where $z, z' \in Z$.

But, elimination of weakly dominated strategies in $G$ may eliminate the $SPE$ in $\Gamma$ (OR 6.6.1).