University of California – Berkeley
Department of Economics
Game Theory in the Social Sciences
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Lecture VI
Food for thought, review of the main ideas, maximizations and evolutionary stability

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Food for thought

LUPI

Many players simultaneously chose an integer between 1 and 99,999. Whoever chooses the lowest unique positive integer (LUPI) wins.

Question What does an equilibrium model of behavior predict in this game?

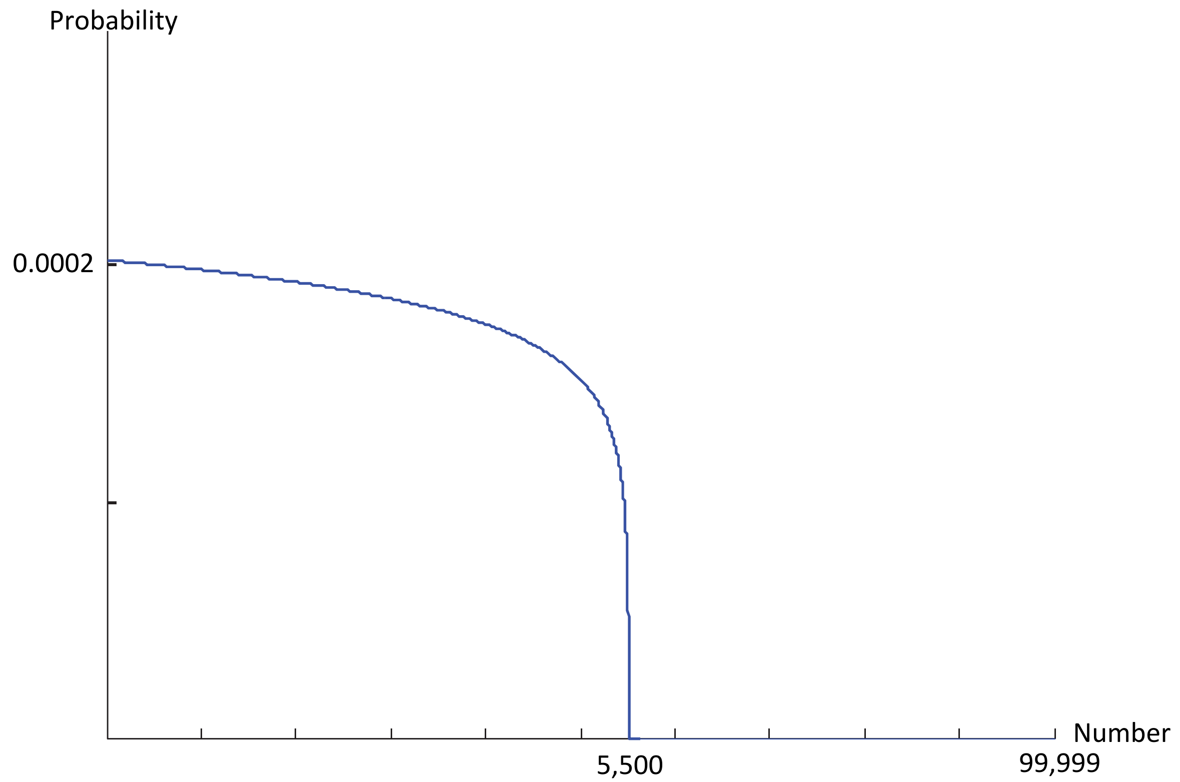
The field version of LUPI, called Limbo, was introduced by the government-owned Swedish gambling monopoly Svenska Spel. Despite its complexity, there is a surprising degree of convergence toward equilibrium.

Games with population uncertainty relax the assumption that the exact number of players is common knowledge.

In particular, in a Poisson game (Myerson; 1998, 2000) the number of players N is a random variable that follows a Poisson distribution with mean n so the probability that $N = k$ is given by

$$\frac{e^{-n}n^k}{k!}$$

In the Swedish game the average number of players was $n = 53,783$ and number choices were positive integers up to $99,999$.



Probability
0.0002
Number
5,500
99,999

Morra

A two-player game in which each player simultaneously hold either one or two fingers and each guesses the total number of fingers held up.

If exactly one player guesses correctly, then the other player pays her the amount of her guess.

Question Model the situation as a strategic game and describe the equilibrium model of behavior predict in this game.

The game was played in ancient Rome, where it was known as “micatio.”

In Morra there are two players, each of whom has four (relevant) actions, S_1G_2, S_1G_3, S_2G_3, and S_2G_4, where S_iG_j denotes the strategy (Show i, Guess j).

The payoffs in the game are as follows

	S_1G_2	S_1G_3	S_2G_3	S_2G_4
S_1G_2	$0, 0$	$2, -2$	$-3, 3$	$0, 0$
S_1G_3	$-2, 2$	$0, 0$	$0, 0$	$3, -3$
S_2G_3	$3, -3$	$0, 0$	$0, 0$	$-4, 4$
S_2G_4	$0, 0$	$-3, 3$	$4, -4$	$0, 0$

Maximal game
(sealed-bid second-price auction)

Two bidders, each of whom privately observes a signal X_i that is independent and identically distributed (i.i.d.) from a uniform distribution on $[0, 10]$.

Let $X^{\text{max}} = \max\{X_1, X_2\}$ and assume the ex-post common value to the bidders is X^{max}.

Bidders bid in a sealed-bid second-price auction where the highest bidder wins, earns the common value X^{max} and pays the second highest bid.

A review of the main ideas

We study two (out of four) groups of game theoretic models:

[1] Strategic games – all players simultaneously choose their plan of action once and for all.

[2] Extensive games (with perfect information) – players choose sequentially (and fully informed about all previous actions).

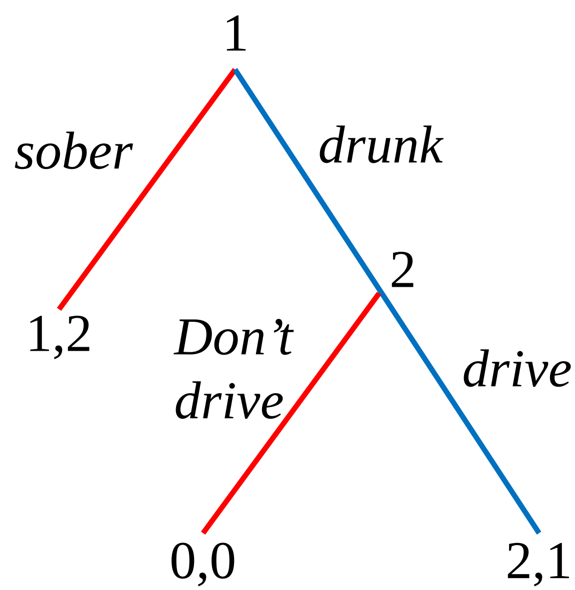
A solution (equilibrium) is a systematic description of the outcomes that may emerge in a family of games. We study two solution concepts:

[1] Nash equilibrium – a steady state of the play of a strategic game (no player has a profitable deviation given the actions of the other players).

[1] Subgame equilibrium – a steady state of the play of an extensive game (a Nash equilibrium in every subgame of the extensive game).

$\Longrightarrow$ Every subgame perfect equilibrium is also a Nash equilibrium.

"LORETTA'S DRIVING BECAUSE I'M DRINKING, AND I'M DRINKING BECAUSE SHE'S DRIVING."


1
sober
drunk
2
1,2
Don't
drive
drive
0,0
2,1

Maximization and zero-sum games

Strictly competitive game

A strategic game $\langle \{1,2\}, (A_i), (\succsim_i) \rangle$ is strictly competitive if for any $a \in A$ and $b \in A$ we have $a \succsim_1 b$ if and only if $b \succsim_2 a$.

	L	R
T	$A, -A$	$B, -B$
B	$C, -C$	$D, -D$

If (x^*, y^*) is a NE of a strictly competitive game then

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y).$$

Maxminimization
(optional)

A $\max\min$ mixed strategy of player i is a mixed strategy that solves the problem

$$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i})$$

A player's payoff in $\alpha^* \in NE(G)$ is at least her $\max\min$ payoff:

$$\begin{aligned} U_i(\alpha^*) &\geq U_i(\alpha_i, \alpha^*_{-i}) \\ &\geq \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \\ &\geq \max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \end{aligned}$$

and the last step follows since the above holds for all $\alpha_i \in \triangle(A_i)$.

Two min-max results

[1] $$\max_{\alpha_i \in \Delta A_i} \min_{\alpha_{-i} \in \Delta A_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{\alpha_i \in \Delta A_i} U_i(\alpha_i, \alpha_{-i})$$

For every α'

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq U_i(\alpha'_i, \alpha'_{-i})$$

and thus

$$\min_{\alpha_{-i}} U_i(\alpha'_i, \alpha_{-i}) \leq \max_{\alpha_i} U_i(\alpha_i, \alpha'_{-i})$$

However, since the above holds for every α'_i and α'_{-i} it must hold for the "best" and "worst" such choices

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i}} \max_{\alpha_i} U_i(\alpha_i, \alpha_{-i}).$$

[2] In a zero-sum game

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha^*)$$

$\Leftarrow$ Since $\alpha^* \in NE(G)$

$$U_1(\alpha^*) = \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^*) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

and since $U_1 = -U_2$ at the same time

$$U_1(\alpha^*) = \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^*, \alpha_2) \leq \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2)$$

Hence,

$$\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) \geq \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2)$$

which together with [1] gives the desired conclusion.

$\Rightarrow$ Let $\alpha_1^{\max}$ be player 1's $\max\min$ strategy and $\alpha_2^{\min}$ be player 2's $\min\max$ strategy. Then,

$$\begin{aligned}\max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1^{\max}, \alpha_2) \\ &\leq U_1(\alpha_1^{\max}, \alpha_2) \ \forall \alpha_2 \in \Delta A_2\end{aligned}$$

and

$$\begin{aligned}\min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) &= \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2^{\min}) \\ &\geq U_1(\alpha_1, \alpha_2^{\min}) \ \forall \alpha_1 \in \Delta A_1\end{aligned}$$

But

$$\begin{aligned} \max_{\alpha_1 \in \Delta A_1} \min_{\alpha_2 \in \Delta A_2} U_1(\alpha_1, \alpha_2) &= \min_{\alpha_2 \in \Delta A_2} \max_{\alpha_1 \in \Delta A_1} U_1(\alpha_1, \alpha_2) \\ &= U_1(\alpha_1^{\max}, \alpha_2^{\min}) \end{aligned}$$

implies that

$$U_1(\alpha_1, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2^{\min}) \leq U_1(\alpha_1^{\max}, \alpha_2)$$

$\forall \alpha_2 \in \Delta A_2$ and $\forall \alpha_1 \in \Delta A_1$.

Hence, $(\alpha_1^{\max}, \alpha_2^{\min})$ is an equilibrium.

Evolutionary Game Theory

Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player's ability to survive.

ε of players consists of mutants taking action a while others take action a^*.

Evolutionary stable strategy (ESS)

Consider a two-player payoff symmetric game

$$G = \langle \{1, 2\}, (A, A), (u_1, u_2) \rangle$$

where

$$u_1(a_1, a_2) = u_2(a_2, a_1)$$

(players exchanging a_1 and a_2).

$a^* \in A$ is ESS if and only if for any $a \in A$, $a \neq a^*$ and $\varepsilon > 0$ sufficiently small

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied if and only if for any $a \neq a^*$ either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

Three results on ESS

[1] If a^* is an ESS then (a^*, a^*) is a NE.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for ε small enough

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus a^* is not an ESS.

[2] If (a^*, a^*) is a <u>strict</u> NE $(u(a^*, a^*) > u(a, a^*)$ for all $a \in A)$ then a^* is an ESS.

Suppose a^* is not an ESS. Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so (a^*, a^*) can be a NE but not a strict NE.

[3] The two-player two-action game

	a	a'
a	w, w	x, y
a'	y, x	z, z

has a strategy which is ESS.

If $w > y$ or $z > x$ then (a, a) or (a', a') are strict NE, and thus a or a' are ESS.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy NE (α^*, α^*) where

$$\alpha^*(a) = (z - x)/(w - y + z - x)$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$.