University of California – Berkeley
Department of Economics
Game Theory in the Social Sciences
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Lecture VII
Repeated Games

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The infinitely repeated prisoner's dilemma

The prisoner's dilemma game with one-shot payoffs

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

has a unique Nash equilibrium in which each player chooses D (defection), but both player are better if they choose C (cooperation).

If the game is played repeatedly, then (C, C) accrues in every period if each player believes that choosing D will end cooperation (D, D), and subsequent losses outweigh the immediate gain.

Strategies

Grim trigger strategy

$$\boxed{\mathcal{S}_0 : C} \xrightarrow[(\cdot, D)]{} \boxed{\mathcal{S}_1 : D}$$

Limited punishment

$$\dashrightarrow \boxed{\mathcal{S}_0 : C} \xrightarrow[(\cdot, D)]{} \boxed{\mathcal{S}_1 : D} \xrightarrow[(\cdot, \cdot)]{} \boxed{\mathcal{S}_2 : D} \xrightarrow[(\cdot, \cdot)]{} \boxed{\mathcal{S}_3 : D} \underset{(\cdot, \cdot)}{\dashrightarrow}$$

Tit-for-tat

$$\dashrightarrow \boxed{\mathcal{S}_0 : D} \xrightarrow[(\cdot, D)]{} \boxed{\mathcal{S}_1 : D} \underset{(\cdot, C)}{\dashrightarrow}$$

Payoffs

Suppose that each player's preferences over streams $(\omega^1, \omega^2, ...)$ of payoffs are represented by the discounted sum

$$V = \sum_{t=1}^{\infty} \delta^{t-1} \omega^t,$$

where $0 < \delta < 1$.

The discounted sum of stream $(c, c, ...)$ is $c/(1-\delta)$, so a player is indifferent between the two streams if

$$c = (1-\delta)V.$$

Hence, we call $(1-\delta)V$ the discounted average of stream $(\omega^1, \omega^2, ...)$, which represent the same preferences.

Consider

$$
\begin{aligned}
V_T &= c + \delta c + \delta^2 c + \cdots + \delta^T c \\
\delta V_T &= \delta c + \delta^2 c + \delta^3 c + \cdots + \delta^{T+1} c.
\end{aligned}
$$

Then,

$$V_T - \delta V_T = c - \delta^{T+1} c.$$

and so

$$V_T = \frac{1 - \delta^{T+1}}{1 - \delta} c \text{ so } V_\infty = \frac{c}{1 - \delta}.$$

Nash equilibria

Grim trigger strategy

$$(1-\delta)(3+\delta+\delta^2+\cdots) = (1-\delta)\left[3+\frac{\delta}{(1-\delta)}\right] = 3(1-\delta)+\delta$$

Thus, a player cannot increase her payoff by deviating if and only if

$$3(1-\delta)+\delta \leq 2,$$

or $\delta \geq 1/2$.

If $\delta \geq 1/2$, then the strategy pair in which each player's strategy is grim strategy is a Nash equilibrium which generates the outcome (C, C) in every period.

Limited punishment (k periods)

$$(1-\delta)(3+\delta+\delta^2+\cdots+\delta^k) = (1-\delta)\left[3+\delta\frac{(1-\delta^k)}{(1-\delta)}\right] = 3(1-\delta)+\delta(1-\delta^k)$$

Note that after deviating at period t a player should choose D from period $t+1$ through $t+k$.

Thus, a player cannot increase her payoff by deviating if and only if

$$3(1-\delta)+\delta(1-\delta^k) \leq 2(1-\delta^{k+1}).$$

Note that for $k=1$, then no $\delta < 1$ satisfies the inequality.

Tit-for-tat

A deviator's best-reply to tit-for-tat is to alternate between D and C or to always choose D, so tit-for tat is a best-reply to tit-for-tat if and only if

$$(1-\delta)(3+0+3\delta^2+0+\cdots) = (1-\delta)\frac{3}{1-\delta^2} = \frac{3}{1+\delta} \leq 2$$

and

$$(1-\delta)(3+\delta+\delta^2+\cdots) = (1-\delta)\left[3+\frac{\delta}{(1-\delta)}\right] = 3-2\delta \leq 2.$$

Both conditions yield $\delta \geq 1/2$.

Subgame perfect equilibria

Grim trigger strategy

For the Nash equilibria to be subgame perfect, "threats" must be credible: punishing the other player if she deviates must be optimal.

Consider the subgame following the outcome (C, D) in period 1 and suppose player 1 adheres to the grim strategy.

Claim: It is not optimal for player 2 to adhere to his grim strategy in period 2.

If player 2 adheres to the grim strategy, then the outcome in period 2 is (D, C) and (D, D) in every subsequent period, so her discounted average payoff in the subgame is

$$(1-\delta)(0+\delta+\delta^2+\cdots)=\delta,$$

where as her discounted average payoff is 1 if she choose D already in period 2.

But, the "modified" grim trigger strategy for an infinitely repeated prisoner's dilemma

$$\boxed{\mathcal{C} : C} \underset{(\cdot,\cdot)/(C,C)}{\longrightarrow} \boxed{\mathcal{D} : D}$$

is a subgame perfect equilibrium strategy if $\delta \geq 1/2$.

Tit-for-tat

The optimality of tit-for-tat after histories ending in (C, C) is covered by our analysis of Nash equilibrium.

If both players adhere to tit-for-tat after histories ending in (C, D): then the outcome alternates between (D, C) and (C, D).

(The analysis is the same for histories ending in (D, C), except that the roles of the players are reversed.)

Then, player 1's discounted average payoff in the subgame is

$$(1-\delta)(3+3\delta^2+3\delta^4+\cdots) = \frac{3}{1+\delta},$$

and player 2's discounted average payoff in the subgame is

$$(1-\delta)(3\delta+3\delta^3+3\delta^5+\cdots) = \frac{3\delta}{1+\delta}.$$

Next, we check if tit-for-tat satisfies the one-deviation property of subgame perfection.

If player 1 (2) chooses C (D) in the first period of the subgame, and subsequently adheres to tit-for-tat, then the outcome is (C, C) $((D, D))$ in every subsequent period. Such a deviation is profitable for player 1 (2) if and only if

$$\frac{3}{(1+\delta)} \geq 2, \text{ or } \delta \leq 1/2$$

and

$$\frac{3\delta}{(1+\delta)} \geq 1, \text{ or } \delta \geq 1/2,$$

respectively.

Finally, after histories ending in (D, D), if both players adhere to tit-for-tat, then the outcome is (D, D) in every subsequent period.

On the other hand, if either player deviates to C, then the outcome alternates between (D, C) and (C, D) (see above).

Thus, a pair of tit-for-tat strategies is a subgame perfect equilibrium if and only if $\delta = 1/2$.