University of California – Berkeley
Department of Economics
ECON 201A Economic Theory
Choice Theory
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Classic demand theory
Part 2 (with derivatives)
(Kreps Ch. 11)

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The (simplest form of the) envelope theorem

Let $f(x, a)$ be a function where x is a choice variable and a is a constraint (determined outside the problem being studied).

– Suppose x is chosen to maximize f and let $x(a)$ be the optimal choice of x for each value of a.

– The (optimal) value function

$$M(a) \equiv f(x(a), a)$$

tells us what the optimized value of f is for each a.

– Differentiating both sides of this identity with respect to a

$$\frac{\partial M}{\partial a} = \frac{\partial f(x(a),a)}{\partial x}\frac{\partial x(a)}{\partial a} + \frac{\partial f(x(a),a)}{\partial a}.$$

Since $x(a)$ is the choice of x that maximizes f, we know that

$$\frac{\partial f(x(a),a)}{\partial x} = 0,$$

and thus

$$\frac{\partial M}{\partial a} = \frac{\partial f(x(a),a)}{\partial a} = \left.\frac{\partial f(x,a)}{\partial a}\right|_{x=x(a)}.$$

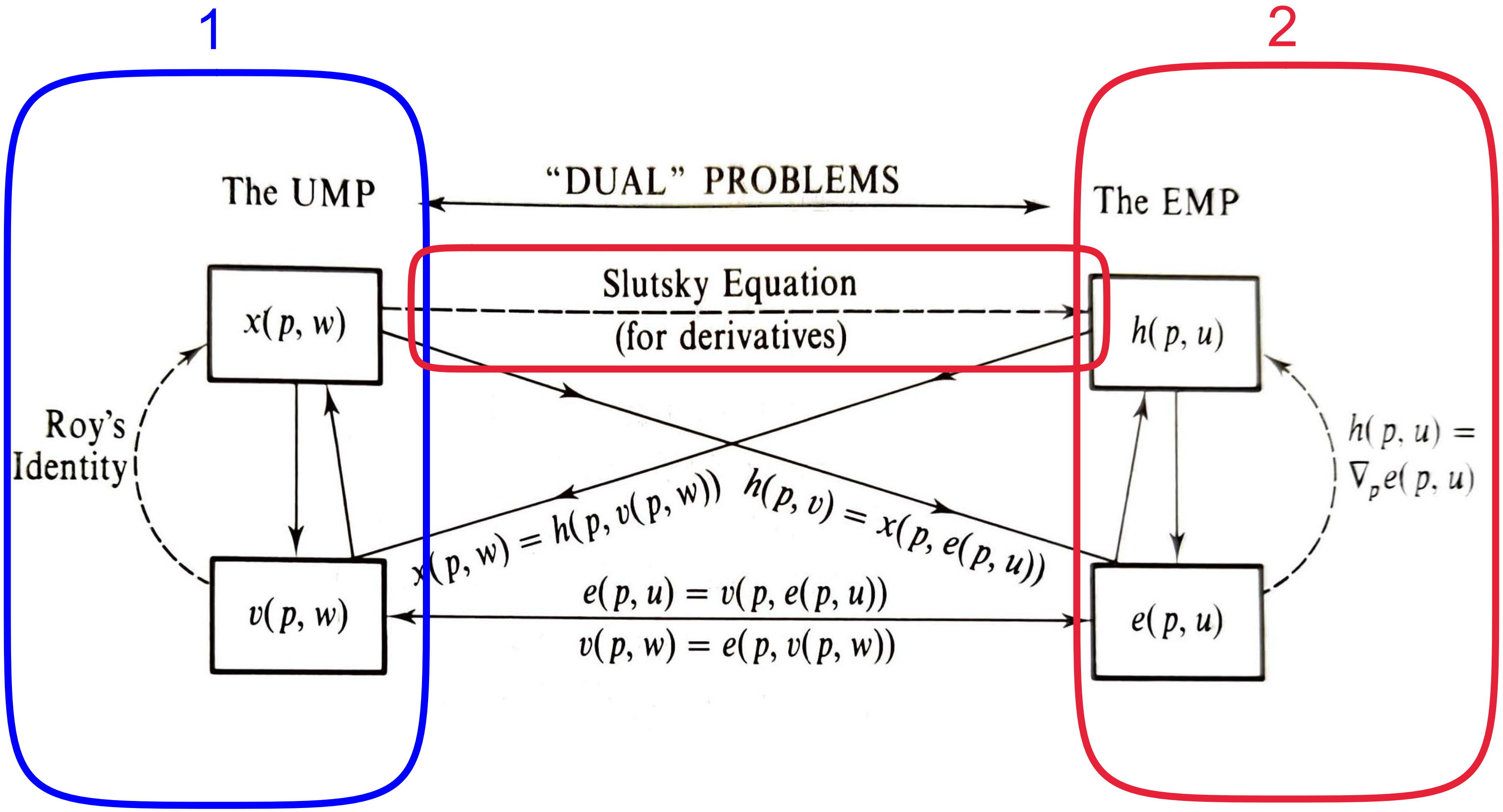
In words, the total derivative of the value function $M(a)$ with respect to the parameter a is equal to the partial derivative evaluated at the optimal choice.

Why? When a changes, there are two effects:

$$\begin{array}{ll} \text{direct:} & a \Longrightarrow f \\ \text{indirect:} & a \Longrightarrow x \Longrightarrow f \end{array}$$

But if x is chosen optimally, then a small change in x has zero effect on f (so the indirect effect drops).

! The conclusion with any number of variables and parameters is similar, but the Lagrange multipliers play an important role...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The UMP
"DUAL" PROBLEMS
The EMP
Slutsky Equation
(for derivatives)
x(p, w)
h(p, u)
Roy's
Identity
h(p, u) =
∇_p e(p, u)
x(p, w) = h(p, v(p, w))
h(p, v) = x(p, e(p, u))
v(p, w)
e(p, u)
e(p, u) = v(p, e(p, u))
v(p, w) = e(p, v(p, w))

Another look at the standard approach for solving the UMP

(i) assume that u is differentiable, (ii) form a Lagrangian, (iii) use the combined 1^{st}-order and complementary slackness conditions.

- no "natural" conditions on $\succsim$ that would guarantee that even well-behaved $\succsim$ admits a differentiable u-representation.

- u-representations of standard $\succsim$ are not continuous so there is no hope that every u-representation of a given $\succsim$ will be differentiable.

Even granting differentiability, we can still ask for the status of the 1^{st}-order and complementary slackness conditions for a non-concave u (more below).

Letting $\lambda \geq 0$ be the multiplier on the budget constraint $p \cdot x \leq \omega$ and $\mu_j \geq 0$ the multiplier on the constraint $x_j \geq 0$, the Lagrangian is

$$u(x) + \lambda \left[\omega - \sum_{j=1}^{K} p_j x_j \right] + \sum_{j=1}^{K} \mu_j x_j$$

The 1st-order conditions (FOC) are

$$\frac{\partial u}{\partial x_j} = \lambda p_j - \mu_j$$

and the complementary slackness conditions

$$\lambda(\omega - p \cdot x) = 0 \text{ and } \mu_j x_j = 0 \text{ for all } j = 1, ..., K$$

must also hold.

Since $\mu_j \geq 0$ these multipliers can be eliminated and the FOC for x_j and the complementary slackness condition for μ_j can be combined as follows

$$\frac{\partial u}{\partial x_j} \leq \lambda p_j \text{ and with } = \text{ if } x_j > 0.$$

And if prices are all strictly positive (which we assume throughout) we can rewrite this as

$$\frac{1}{p_j}\frac{\partial u}{\partial x_j} \leq \lambda \text{ and with } = \text{ if } x_j > 0.$$

$\Rightarrow$ For goods $x_j > 0$, the ratios of the marginal utility of the goods to their respective prices must be equal (and greater than the corresponding ratios for goods $x_j = 0$).

If $\lambda > 0$, we get the MRS of good i for good j (along an indifference curve) equals the ratio of their prices (intermediate micro):

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{p_i}{p_j}.$$

But is $\lambda > 0$? Because $v(p, \omega) = \max\{u(x) : p \cdot x \leq \omega \text{ and } x \geq 0\}$ then by constrained optimization, $\partial v/\partial \omega = \lambda$.

- v is strictly increasing in ω, but there are strictly increasing, differentiable functions whose derivatives are zero at isolated points...

- even if u represents $\succsim$ convex preferences, there are strictly increasing and quasi-concave functions whose derivatives go to zero at points...

– if u is concave then $v(p,\omega)$ is concave in ω (verify this!), and a strictly increasing, concave function cannot have zero derivative.

– convexity of $\succsim$ only guarantees that it admits a quasi-concave u-representation so concavity of u is hard to do on first principles.

– Concavity of u is not necessarily preserved by monotonic transformations, so we cannot guarantee that every representation of $\succsim$ is concave.

$\Rightarrow$ Even strictly increasing and strictly convex $\succsim$ cannot guarantee that $\lambda > 0$ because the u-representation is 'only' quasi-concave.

Two (serious) assumptions

1. $\succsim$ are strictly convex so UMP and EMP have unique solutions for every (p, ω) and (p, u).

2. $x(p, \omega)$, $h(p, u)$, $v(p, \omega)$ and $h(p, u)$ are all continuously differentiable functions of all their arguments.

Note 1 UMP and EMP have unique solutions + standard techniques $\Rightarrow$ the four functions are all continuous.

Note 2 Differentiability requires a lot more: u is twice-continuously differentiable (C^2) and well-behaved along axes...

Back to the expenditure-minimization problem (EMP)

$e(p,u)$ – the inverse of $v(p,\omega)$ – indicates the minimal income ω needed to achieve utility level u at prices p:

$$e(p,u) = \min_x p \cdot x$$
$$\text{subject to } u(x) \geq u.$$

$e(p,u)$ and the Hicksian demand $h(p,u)$ are related as follows:

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}. \qquad (*)$$

Proof: Differentiate both sides of the accounting identity $e(p,u) = p \cdot h(p,u)$ with respect to p_i:

$$\frac{\partial e}{\partial p_i} = h_i(p,u) + \sum_{j=1}^{K} p_j \frac{\partial h_j}{\partial p_i}.$$

In words, if $p_i \uparrow$ then the resulting change in expenditure needed to reach utility level u comes from two terms:

1. The amount h_i of good i bought is more expensive and expenditure rises at the rate $h_i(p,u)$.

2. The "cost" of changes in the optimal bundle (less i and more or less $j \neq i$) given by the sum term.

The result we are supposed to be heading for says that the summation term is zero...

– If η is the Lagrange multiplier on the constraint, the FOC (for x_i) of the EMP is given by

$$p_i = \eta \frac{\partial u}{\partial x_i}, \tag{**}$$

evaluated at the optimum $h(p, u)$.

– Implicitly partially differentiate both sides of the accounting identity $u\left(h(p, u)\right) = u$ with respect to p_i

$$\sum_{j=1}^{K} \frac{\partial u}{\partial x_j} \frac{\partial h_j}{\partial p_i} = 0. \tag{***}$$

– $(**)$ and $(***)$ yield

$$\frac{1}{\eta}\sum_{j=1}^{K} p_j \frac{\partial h_j}{\partial p_i} = 0,$$

which is just what we want, as long η is not zero (or infinite).■

Note 1 The 'technique' is to substitute into one equation a FOC for something that is optimal in a constrained optimization problem.

Note 2 This technique is formalized in the envelope theorem because $e(p, u)$ is the 'lower envelope' of linear functions (like the one we will draw next).

A 'slick' graphical proof: Fix the utility argument u^* and all the prices p_j^* except for p_i and graph the function

$$p_i \longrightarrow e((p_i, p_{-i}^*), u^*),$$

which is a concave function in p_i (and assume it is also differentiable...).

Since the utility from the bundle $h(p^*, u^*)$ is u^*

$$e((p_i, p_{-i}^*), u^*) \leq p_i h_i(p^*, u^*) + \sum_{j \neq i} p_j^* h_j(p^*, u^*)$$

and with $=$ if $p_i = p_i^*$. The RHS is a linear function of p_i, and its slope in the p_i direction is $h_i(p^*, u^*)$.■

Back to Roy's identity...

$x(p,\omega)$ and $v(p,\omega)$ are related as follows: $x_i(p,\omega) = -\dfrac{\partial v/\partial p_i}{\partial v/\partial \omega}$.

Proof: Suppose $x^* = x(p,\omega)$ and let $u^* = u(x^*)$. Using the above identities (and assuming that UMP and EMP have unique solutions)

$$x^* = h(p,u^*) \text{ and } \omega = e(p,u^*)$$

so $u^* = v(p, e(p,u^*))$ for a fixed u^* and for all p. Differentiating this with respect to p_i

$$0 = \frac{\partial v}{\partial p_i} + \frac{\partial v}{\partial \omega}\frac{\partial e}{\partial p_i},$$

replace $\dfrac{\partial e}{\partial p_i}$ with $h_i(p,u^*) = x_i^* = x_i(p,\omega)$, and rearrange.■

To demystify Roy's identity, rewrite it as

$$-\frac{\partial v}{\partial p_i} = \frac{\partial v}{\partial \omega} x_i$$

If $x_i(p, \omega) > 0$ then the FOC for x_i in the UMP can be written as

$$\frac{1}{p_i} \frac{\partial u}{\partial x_i} = \lambda$$

and since $\partial v / \partial \omega = \lambda$ in the UMP, we can substitute and combine

$$-\frac{\partial v}{\partial p_i} = \lambda x_i = \frac{x_i}{p_i} \frac{\partial u}{\partial x_i}.$$

The consumer uses the "extra" income when $p_i \downarrow$ naively, spending all of it on x_i which raises u by

$$\frac{x_i}{p_i}\frac{\partial u}{\partial x_i}$$

– The consumer can (and probably will) do some further substituting among all good.

– But Roy's identity tells us that these substations will not have a 1$^{\text{st}}$-order effect on utility at the optimum (draw a picture!).

The Slutsky equation – connecting $x(p, \omega)$ and $h(p, u)$

Question (from intermediate micro or even a good principles course) starting at p and ω, what happens to $x_j(p, \omega)$ if $p_i \uparrow$?

[1] The general "price index" $\uparrow$ so the consumer is a bit poorer (in real terms) $\Longrightarrow$ change the demand $x_j(p, \omega)$ (roughly) at a rate

$$-\frac{\partial x_j}{\partial \omega} x_i(p, \omega)$$

(\$0.01 rise in p_i means $x_i(p, \omega) \times 0.01$ less to spend).

[2] There is also a "cross-substitution" $x_j(p, \omega)$: (probably) $x_i \downarrow$ and depending on the relationship between i and j, $x_j \uparrow$ or $\downarrow$.

There are two (obvious) ways to compensate our poorer consumer:

- Slutsky: $\uparrow \omega$ (just) enough $\Longrightarrow$ the consumer could afford the bundle consumed.

- Hicks: $\uparrow \omega$ (just) enough $\Longrightarrow$ the consumer will be as well off (after re-optimizing).

! The Hicksian compensation is a theoretical construct – depends on unobservable $\succsim$ – but since we have the Hicksian demand function $h(p, u)$ defined it is simply $\partial h_j / \partial p_i$.

Slutsky equation: $x(p,\omega)$ and $h(p,u)$ are are related as follows:

$$\frac{\partial x_j}{\partial p_i} = \underbrace{\frac{\partial h_j}{\partial p_i}}_{\substack{\text{Hicksian} \\ \text{compensation}}} - \underbrace{\frac{\partial x_j}{\partial \omega} x_i}_{\substack{\text{income} \\ \text{adjustment}}}$$

evaluated for given p and ω and $x(p,\omega)$ and $h(p,u)$ where u is the utility level achieved at $x(p,\omega)$.

! We·cannot know if the Hicksian compensation is correct, and our income adjustment is not quite correct because of the substitution out of x_i (as discussed above).

Proof: Differentiate both sides of the identity $x_j(p, e(p, u)) = h_j(p, u)$ with respect to p_i

$$\frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial \omega}\frac{\partial e}{\partial p_i} = \frac{\partial h_j}{\partial p_i}$$

but since $\dfrac{\partial e}{\partial p_i} = h_i(p, u) = x_i(p, e(p, u)$, we get

$$\frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial \omega} x_i(p, e(p, u) = \frac{\partial h_j}{\partial p_i}. \blacksquare$$

“...you may be wondering where all this is headed. We are certainly taking lots of derivatives, and it isn’t at all clear to what end we are doing so. Some results are coming, but we need a bit more setting-up to get them. Please be patient...” – Kreps –

A mathematical fact of twice-continuously differentiable (C^2) concave functions

For a given C^2 function $f : \mathbb{R}^K \to \mathbb{R}$ and any $z \in \mathbb{R}^K$, let $H(z)$ be $K \times K$ matrix whose $(i, j)^{\text{th}}$ element is

$$\left.\frac{\partial^2 f}{\partial z_i \partial z_j}\right|_z$$

(mixed second partials of f, evaluated at z).

This matrix is called the <u>Hessian matrix</u> of f and it is automatically symmetric.

- A C^2 function of one variable $f : \mathbb{R} \longrightarrow \mathbb{R}$ is concave if its derivative is non-increasing (2nd-derivative is non-positive).

For concave functions of several variables this generalizes as follows:

- A C^2 function $f : \mathbb{R}^K \longrightarrow \mathbb{R}$ is concave iff its Hessian matrix (evaluated at each point in the domain of f) is negative semi-definite.

- If H is a negative semi-definite $K \times K$ matrix then $H_{ii} \leq 0$ for all $i = 1, ..., K$.

The main result(s) – connecting all the pieces

The $K \times K$ matrix of substitution terms whose $(i, j)^{\text{th}}$ element is

$$\frac{\partial x_i(p, \omega)}{\partial p_j} + \frac{\partial x_i(p, \omega)}{\partial \omega} x_j(p, \omega)$$

is symmetric and negative semi-definite.

<u>Proof</u>: By the Slutsky equation, $(i, j)^{\text{th}}$ element is

$$\frac{\partial h_i(p, u)}{\partial p_j}$$

evaluated at $(p, u(x(p, \omega)))$ and by $(\not\ast)$

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j}.$$

Integrability

We concluded that UMP (equivalently, EMP) imposes that the matrix of substitution terms

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i(p,\omega)}{\partial p_j} + \frac{\partial x_i(p,\omega)}{\partial \omega} x_j(p,\omega)$$

must be symmetric and negative semi-definite.

Question (the integrability problem) suppose there is a system of $x_i(p,\omega)$ that have a symmetric and negative semi-definite substitution terms matrix, is there necessarily a u-maximizing consumer behind it?!

As we know $x(p, \omega)$ should be (i) homogeneous of degree zero and (ii) should obey Walras' law with equality.

$$(i) + (ii) + \text{symmetric and negative semi-definite substitution matrix}$$
$$\Downarrow$$
$$\text{substitution matrix can be ``integrated up'' to get a representative } v(p, \omega)$$

We will not even attempt to sketch the proof here — requires to determine a solution of a system of partial differential equations.

Last word: aggregate consumer demand

It is hard (not impossible!) to obtain individual-level data:

- Aggregate demand will be homogeneous of degree zero in prices and (total) income.
- Walras' law will hold for the entire economy, if all consumers are locally-insatiable,

But results analogous to the Slutsky restrictions (or GA.RP) do not generally hold for aggregate demand.

$\Longrightarrow$ Make strong assumptions about the distribution of preferences/income, e.g. the same homothetic preferences.