University of California – Berkeley
Department of Economics
ECON 201A Economic Theory
Choice Theory
Fall 2023

Choice under uncertainty
Part I: von Neumann-Morgenstern expected utility
(Kreps Ch. 5 and Rubinstein Ch. 7)

Oct 3, 2023

Formulation and representation

We seek to expand upon the story we told so far, where we had a binary (preference) relation $\succ$ on a set X and we sought a function $u : X \to \mathbb{R}$ representing $\succ$ in the sense of

$$x \succ y \Leftrightarrow u(x) > u(y).$$

This is just as before if we assume that $\succ$ is

- <u>asymmetric</u>: there is no pair $x, y \in X$ such that $x \succ y$ and $x \prec y$.

- <u>negatively transitive</u>: for any n, if $x \succ y$ then for any $z \in X$ either $x \succ z$ or $z \succ y$, or both.

If $\succ$ is asymmetric and negatively transitive then it is

- <u>irreflexive</u>: $x \succ x$ for no $x \in X$.

- <u>transitive</u>: $x \succ y$ and $y \succ z \Rightarrow x \succ z$.

- <u>acyclic</u>: $x_1 \succ x_2, x_2 \succ x_3, ..., x_{n-1} \succ x_n \Rightarrow x_1 \neq x_n$.

! We began with $\succsim$ and induce $\succ$ and $\sim$. We now take $\succ$ as primitive (what the $\mathcal{DM}$ expresses) but it makes (almost) no difference and we can construct $\succsim$ and $\sim$ also as before.

The goal: X to represent uncertain prospects and to 'specialize' the form of the u-function representing $\succ$ by imposing further conditions on $\succ$ based on the (mathematical) structure of X.

Question How (mathematically) do we model an uncertain prospect and what corresponding forms of functions u should we seek?

The literature contains (basically) three sets of answers to these questions, differing in whether uncertainty is objective or subjective.

(1) and (2) are polar cases and (3) is a middle case:

1. Objective uncertainty: von Neumann-Morgenstern (vNM).
2. Subjective uncertainty: Savage.
3. Horse lottery-roulette wheel theory: Anscombe and Aumann (A-A).

In A-A, the $\mathcal{DM}$ is assumed to have some objective randomizing devices that s/he can employ, e.g. fair coins, color wheels, roulette wheels...

! The point is to understand how these models differ as representations of uncertain prospects and to think why one might be a more appropriate model...

vNM expected utility
(with finite prize spaces)

In the vNM model uncertainty is objective – there is given a quantification of how likely the various outcomes are (given in the form of a probability distribution).

– some given arbitrary set X of prizes/consequences

– a set P of probability measures or probability distributions on X.

P is the choice set – the $\mathcal{DM}$ is choosing/expressing preference among probability distributions.

We take on the easiest case, where the set of possible prizes X is a finite set and P is the set of functions $P : X \longrightarrow [0, 1]$ such that

$$\sum_{x \in X} p(x) = 1$$

The $\mathcal{DM}$ is presumed to be making pairwise comparisons between members of P, indicating strict preference by the binary relation $\succ$.

! If X is countably infinite, P can still be defined as above. But when X is infinite (countable or not) P can be the set of all simple or discrete (or more complicated) probability measures (mixture-space theorem).

A compound lottery: If $p, q \in P$ and $\alpha \in [0,1]$ then there is an element $\alpha p + (1-\alpha)q \in P$ which is defined by taking the convex combinations of the probabilities of each prize separately, or

$$(\alpha p + (1-\alpha)q)(x) = \alpha p(x) + (1-\alpha)q(x)$$

so $\alpha p + (1-\alpha)q$ represents a compound lottery.

$\Rightarrow$ Can the $\mathcal{DM}$ apply the expected-utility model "locally" to an immediate problem, abstracting away from the other considerations, sources of uncertainty, and more? (Rabin's critique/paradox)

Three (or more) axioms
(about $\succ$ on P)

(A1) $\succ$ is a preference relation (asymmetric and negatively transitive)

- Negative transitivity is troublesome and the richer setting does not make it any less so, but let's not worry further about it...

(A2) For all $p, q, r \in P$ and $\alpha \in [0, 1]$, $p \succ q \Rightarrow \alpha p + (1 - \alpha) r \succ \alpha q + (1 - \alpha) r$.

- This is the independence (or substitution) axiom, a straightforward and compelling normative precept for choice under uncertainty.

(A3) For all $p, q, r \in P$, if $p \succ q \succ r$ then there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha p + (1 - \alpha) r \succ q \succ \beta p + (1 - \beta) r.$$

- This is called the Archimedean or continuity axiom – 'resemblance' to the Archimedes' principle:

  for all $0 < x < y$ there is (an integer) n such that $nx > y$.

- The reason that it is also called the continuity axiom will become clear in a little bit...

- What if p, q and r are (respectively) \$1000, \$10 and death (for sure)?!

Regardless of how you feel about (A1)-(A3), together these axioms yield the following result:

Theorem (vNM): $\succ$ on P satisfies axioms (A1)-(A3) if and only if there exists a function $u : X \rightarrow \mathbb{R}$ such that

$$p \succ q \Leftrightarrow \sum_x p(x)u(x) > \sum_x q(x)u(x). \qquad (*)$$

Moreover, u is unique up to a positive affine transformation: if u represents $\succ$ in the sense of $(*)$, then $u' : X \rightarrow \mathbb{R}$ also represents $\succ$ if and only if there exist real numbers $c > 0$ and d such that

$$u'(\cdot) = cu(\cdot) + d.$$

Two remarks:

(1) u-representations are unique up to strictly increasing rescalings: if u represents $\succ$ then so will $v(\cdot) = f(u(\cdot))$ for any strictly increasing f.

But if f is an arbitrary increasing function, then the v that results from composing u with f may not have the expected-utility form.

(2) Continuing on this general point, we said that there is no cardinal significance in utility differences.

But in the context of expected utility where u gives an expected-utility representation on P, utility differences have cardinal significance.

To illustrate this point, suppose

$$u(x) - u(x'') = 2(u(x') - u(x'')) > 0$$

which does not mean that x is twice better than x'' than is x' – it just means that $x \succ x' \succ x''$.

But when u gives an expected-utility representation on P, utility differences have cardinal significance

$$p := \begin{array}{ll} \overset{1/2}{\nearrow} & x \\ & \\ \underset{1/2}{\searrow} & x'' \end{array} \quad \sim \quad q := \xrightarrow{1} x'$$

Finally, note that if we define $f : P \longrightarrow \mathbb{R}$ by

$$f(p) = \sum_x p(x)u(x)$$

then $(*)$ becomes $p \succ q \Leftrightarrow f(p) > f(q)$ so f gives an ordinal representation of $\succ$ (in the standard sense).

Note also that P is uncountable. But we know (?) there is a countable $\succ$ order dense subset of P (implied by the axioms, mostly (A3)).

Three lemmas

How is the vNM theorem proven? We first use (A1)-(A3) to obtain three lemmas...

If $\succ$ on P satisfies (A1)-(A3) then:

(L1) $p \succ q$ and $0 \leq \alpha < \beta \leq 1 \Rightarrow \beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q$.

- In words, if we look at (binary) compounded lotteries, the $\mathcal{DM}$ always (strictly) prefers a higher probability of "winning" the preferred lottery.

(L2) $p \succsim q \succsim r$ and $p \succ r \Rightarrow$ there exists a <u>unique</u> $\alpha^* \in [0, 1]$ such that

$$q \sim \alpha^* p + (1 - \alpha^*) r.$$

– This result (sometimes simply assumed) is called the calibration property – calibrate the $\mathcal{DM}$'s preference for any lottery in terms of a lottery that involves only the best and worst prizes.

– By virtue of (L1), we know there is exactly one value α^* that will do in (L2). This is what causes (A3) to be called a continuity axiom – the preference ordering is continuous in probability.

(L3) $p \sim q$ and $\alpha \in [0,1] \Rightarrow$ for all $r \in P$

$$\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r.$$

– This is just like the independence axiom (A2), except that $\succ$ is replaced by $\sim$. This is sometimes assumed as an axiom, and it itself is then sometimes called the substitution axiom.

A sketch of the lemmas' proofs

(L1) $p \succ q$ and $0 \le \alpha < \beta \le 1 \Rightarrow \beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q$.

Proof: If $\alpha = 0$ then $p \succ q$ and $0 < \beta \le 1$ with (A2) imply

$$\beta p + (1-\beta)q \succ \beta q + (1-\beta)q = q = \alpha p + (1-\alpha)q.$$

Let $r = \beta p + (1-\beta)q$ and suppose $\alpha > 0$. Then $\dfrac{\alpha}{\beta} < 1$, and $r \succ q$ and (5.2) imply

$$\begin{aligned} r &= (1-\frac{\alpha}{\beta})r + \frac{\alpha}{\beta}r \\ &\succ (1-\frac{\alpha}{\beta})q + \frac{\alpha}{\beta}r \\ &= (1-\frac{\alpha}{\beta})q + \frac{\alpha}{\beta}(\beta p + (1-\beta)q) \\ &= \alpha p + (1-\alpha)q. \blacksquare \end{aligned}$$

(L2) $p \succsim q \succsim r$ and $p \succ r \Rightarrow$ there exists a unique $\alpha^* \in [0,1]$ such that

$$q \sim \alpha^* p + (1-\alpha^*)r.$$

Proof: Since $p \succ r$, (L1) ensures that if α^* exists it is unique. If $p \sim q$ (resp. $q \sim r$) then $\alpha^* = 1$ (resp. $\alpha^* = 0$) works.

Hence, we only need to consider the case $p \succ q \succ r$. Define

$$\alpha^* = \sup\{\alpha \in [0,1] : q \succsim \alpha p + (1-\alpha)r]$$

(since $\alpha = 0$ is in the set, we are not tacking a sup over an empty set...).

Assuming,

$$\alpha^* p + (1 - \alpha^*) r \succ q \succ r$$

or

$$p \succ q \succ \alpha^* p + (1 - \alpha^*) r$$

leads to a contradiction by (A3) (verify this!), which leaves us with the third possibility

$$q \sim \alpha^* p + (1 - \alpha^*) r$$

(which is what we want).■

(L3) $p \sim q$ and $\alpha \in [0, 1] \Rightarrow$ for all $r \in P$

$$\alpha p + (1 - \alpha) r \sim \alpha q + (1 - \alpha) r.$$

Proof: Suppose that there is some $s \in P$ with $s \succ p \sim q$ (otherwise, trivial) and that $\alpha p + (1 - \alpha) r \succ \alpha q + (1 - \alpha) r$ toward contradiction.

(A2) implies that for all $\beta \in (0, 1)$

$$\beta s + (1 - \beta) q \succ \beta q + (1 - \beta) q = q \sim p$$

and (A2) also implies

$$\alpha(\beta s + (1 - \beta) q) + (1 - \alpha) r \succ \alpha p + (1 - \alpha) r.$$

And by since assumption

$$\alpha(\beta s + (1-\beta)q) + (1-\alpha)r \succ \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r\,,$$

(A3) implies that for each β there exists some $\alpha^*(\beta) \in (0,1)$ such that

$$\begin{gathered} \alpha p + (1-\alpha)r \\ \succ \\ \alpha^*(\beta)(\alpha(\beta s + (1-\beta)q) + (1-\alpha)r) \\ + \\ (1-\alpha^*(\beta))(\alpha q + (1-\alpha)r). \end{gathered}$$

Fix, for example, $\beta = 1/2$ and let $\alpha^*(1/2)$ written as α^* then the term on the RHS is

$$[\frac{\alpha^*\alpha}{2}]s + [\frac{\alpha^*\alpha}{2} + (1-\alpha^*)\alpha]q + [1-\alpha]r$$

$$\shortparallel$$

$$\alpha[\frac{\alpha^*}{2}s + (1-\frac{\alpha^*}{2})q] + (1-\alpha)r.$$

But since $\frac{\alpha^*}{2} > 0$, this last term must be $\succ\ \alpha p + (1-\alpha)r$, a contradiction.$\blacksquare$

Another lemma...

Before stating another (final) lemma, we need some notation: For any $x \in X$, let δ_x denote the probability distribution degenerate at x, that is

$$\delta_x(x') = \begin{cases} 1 & \text{if} \quad x' = x \\ 0 & \text{if} \quad x' \neq x \end{cases}$$

(L4) If $\succ$ on P satisfies (A1)-(A3) then for all $p \in P$ there exist $x^\circ, x_\circ \in X$ such that $\delta_{x^\circ} \succsim p \succsim \delta_{x_\circ}$.

The proof (omitted) builds on (A2) and (L3) and uses induction on the size of the support of p.

Proof of the vNM theorem: Showing that if a u-function as in $(*)$ exists then (A1)-(A3) all hold is omitted (straightforward?).

- Suppose that $\succ$ satisfies (A1)-(A3) and use (L4) to produce δ_{x° and $\delta_{x_\circ}$. If $\delta_{x^\circ} \sim \delta_{x_\circ}$ then $u \equiv c$ (constant) satisfies $(*)$ as neither

$$p \succ q \text{ nor } \sum p(x)u(x) > \sum q(x)u(x)$$

is possible.

Also, u constant is the only possible function for this representation in this case, so u' is any other representation if and only if it is a positive affine transformation of u.

– From now on, assume $\delta_{x^\circ} \succ \delta_{x_\circ}$ and for any $p \in P$ define

$$f(p) = \alpha \text{ where } \alpha\delta_{x^\circ} + (1-\alpha)\delta_{x_\circ} \sim p.$$

By the lemmas, such an α exists and is unique, so f is well defined.

By (L1) and standard properties of preference relations

$$\begin{array}{rcl} f(p) & > & f(q) \\ & \Updownarrow & \\ f(p)\delta_{x^\circ} + (1-f(p))\delta_{x_\circ} & \succ & f(q)\delta_{x^\circ} + (1-f(q))\delta_{x_\circ} \\ & \Updownarrow & \\ p & \succ & q. \end{array}$$

– Hence $f(\cdot)$ is a representation of $\succ$ in the standard sense but we are not done quite yet!

We will have the expected-utility representation $(*)$ as soon as we show that for all $p \in P$

$$f(p) = \sum p(x)u(x). \qquad (\dagger)$$

The method is to use $(\dagger\dagger)$ below and induction on the size of the support of p

$$\{x \in X : p(x) > 0\}.$$

– Note that for all $p, q \in P$ and $\alpha \in [0,1]$, by repeated application of (L3)

$$\alpha p + (1-\alpha) q$$
$$\sim$$
$$\alpha[f(p)\delta_{x^\circ} + (1 - f(p))\delta_{x_\circ}] + (1-\alpha)[f(q)\delta_{x^\circ} + (1 - f(q))\delta_{x_\circ}]$$

So by the definition of f

$$f(\alpha p + (1-\alpha) q) = \alpha f(p) + (1-\alpha) f(q). \qquad (\dagger\dagger)$$

– If the support of p has one element, say x', then $p = \delta_{x'}$ and (†) follows trivial. Suppose inductively that (†) is true for p with support of size $n - 1 \geq 1$.

Take any p with support of size $n > 1$, let x' be in the support of p, and defined q as follows:

$$q(x) = \begin{cases} 0 & \text{if} \quad x = x' \\ p(x)/(1 - p(x')) & \text{if} \quad x \neq x' \end{cases}$$

so q has support of size $n - 1$ and $p = p(x')\delta_{x'} + (1 - p(x'))q$.

By (††) and the induction hypothesis applied to q

$$f(p) = p(x')f(\delta_{x'}) + (1 - p(x'))f(q)$$

$$\shortparallel$$

$$p(x')u(x') + (1 - p(x'))\sum_{x \neq x'} \frac{p(x)}{1 - p(x')}u(x)$$

$$\shortparallel$$

$$\sum_x p(x)u(x).$$

This establishes (†) by induction, since X is finite.

! We also need to establish the uniqueness result: if u and u' represent $\succ$ in the sense of $(*)$ then then each is a positive affine transformation of the other.■

Final comments:

- There is an implicit axiom zero: All that matters to the $\mathcal{DM}$ are probabilities and prizes – the randomizing devices and their order are inconsequential.

- The next order of business is to obtain the vNM representation when X is infinite and P is more complicated – concerning a quite abstract object called a mixture space.

- This part of 201A is about choice theory so it ought to have something to say about your choice problem. Does it (so far), despite some of the damning experimental evidence?!