Economics 209A
Theory and Application of Non-Cooperative Games
(Fall 2013)

Supermodular games

Introduction

- Each player's marginal utility of "increasing" his strategy rises with increases of the other players' strategies.

- In such games, the best response correspondences are increasing, so that players' strategies are <u>strategic complements</u>.

- Supermodular games are simple and well-behaved (they have pure strategy Nash equilibrium).

The main ideas

Consider a symmetric n-player game in which $s_i \in [0,1]$ and $\pi(s_i, \bar{s}_{-i})$, where

$$\bar{s}_{-i} \equiv \sum_{j \neq i} \frac{s_j}{n-1}.$$

– $\pi(\cdot)$ exhibits positive spillovers if $\pi(s_i, \bar{s}_{-i})$ is increasing in $\bar{s}_{-i}$.

– $\pi(\cdot)$ exhibits strategic complementarities (increasing first differences) if $\pi(s_i', \bar{s}_{-i}) - \pi(s_i, \bar{s}_{-i})$ is increasing in $\bar{s}_{-i}$ for all $s_i' > s_i$.

– A symmetric Nash equilibrium (SNE) is an action $s^* \in [0,1]$ such that $\pi(s^*, s^*_{-i}) \geq \pi(x, s^*_{-i})$ for all $s \in [0,1]$.

Claim: (Weak) strategic complements are necessary over some range for multiple symmetric Nash equilibrium.

- By contradiction. Suppose that $\pi(\cdot)$ satisfies (strictly) decreasing first differences and that $s^*, s^{**} \in SNE$ s.t. $s^* < s^{**}$

- Then, the equilibrium conditions implies

$$\pi(s^{**}, s^*) - \pi(s^*, s^*) \leq 0$$

  and decreasing first differences implies

$$\pi(s^{**}, s^{**}) - \pi(s^*, s^{**}) < 0$$

  which contradicts the assumption that s^{**} is a SNE.

The strategic-form game

Consider a set N of players, and for each player $i \in N$

- a non-empty set $S_i \subset \mathbb{R}^{\ell_i}$ of actions (not necessarily compact and convex).
- a utility function $u_i : S \to \mathbb{R}$ where $S = \times_{j \in N} S_j \subset \mathbb{R}^{\ell}$ and $\ell \equiv \sum_{i \in N} \ell_i$ is the set of possible outcomes.

Lattices

Let $\mathbb{R}^K$ denote the finite K-dimensional vector space and let $\geq$ denote the usual partial (vector) ordering on $\mathbb{R}^K$, that is, for any $x, y \in \mathbb{R}^K$,

$$x \geq y \Longleftrightarrow x_k \geq y_k$$

for all $k = 1, ..., K$, and we also write $x > y \Longleftrightarrow x \geq y$ and $x \neq y$.

The meet (resp. join) of x and y is denoted by $x \wedge y$ (resp. $x \vee y$) and defined by

$$x \wedge y = (\min(x_1, y_1), ..., \min(x_K, y_K)),$$

and

$$x \vee y = (\max(x_1, y_1), ..., \max(x_K, y_K)).$$

A sub-lattice of a lattice (partially ordered set) L is a nonempty subset of L which is a lattice with the same meet and join operations as L.

S is a <u>sub-lattice</u> of $\mathbb{R}^\ell$ if $s \in S$ and $s' \in S$ implies that $s \wedge s' \in S$ and $s \vee s' \in S$.

The sub-lattice S has a <u>greatest</u> (resp. <u>least</u>) element $\bar{s} \in S$ (resp. $\underline{s} \in S$) if $\bar{s} \geq s$ (resp. $\underline{s} \leq s$) for all $s \in S$.

Increasing differences

The notion of increasing differences formulizes the notion of strategic complementarily:

$u_i(s_i, s_{-i})$ exhibits increasing differences if

$$u_i(s_i', s_{-i}') - u_i(s_i, s_{-i}') \geq u_i(s_i', s_{-i}) - u_i(s_i, s_{-i})$$

whenever

$$s_i' \geq s_i \text{ and } s_{-i}' \geq s_{-i},$$

and exhibits strictly increasing differences when the inequalities are strict.

That is, an increase in the strategies of the other players raises the desirability of playing a higher strategy for player i.

Supermodularity

$u_i(s_i, s_{-i})$ is <u>supermodular</u> in s_i if for each s_{-i}

$$u_i(s_i, s_{-i}) + u_i(s_i', s_{-i}) \leq u_i(s_i \wedge s_i', s_{-i}) + u_i(s_i \vee s_i', s_{-i})$$

for all $s_i, s_i' \in S_i$, and <u>strictly supermodular</u> when the inequalities are strict.

<u>Remark I</u>: supermodularity is always satisfied if s_i and s_i' can be ordered by $\geq$, so the strength of supermodularity applies to cases where s_i and s_i' cannot be so ordered.

Remark II: supermodularity ensures that increasing first differences implies strategic complementarity.

If $u_i(s_i, s_{-i})$ exhibits increasing differences but is not supermodular in s_i, then the best response need not be monotonically non-decreasing in the other players' strategies.

To prove that u_i exhibits increasing differences, let $s'_i \geq s_i$ and $s'_{-i} \geq s_{-i}$, where $s_{-i}, s'_{-i} \in s_{-i}$ and $s_i, s'_i \in s_i$. Let $u = (s_i, s'_{-i})$ and $v = (s'_i, s_{-i})$. Then the definition of supermodularity implies that

$$u_i(u \vee v) + u_i(u \wedge v) \geq u_i(u) + u(v)$$

which can be written

$$u_i(s'_i, s'_{-i}) + u_i(s_i, s_{-i}) \geq u_i(s_i, s'_{-i}) + u(s'_i, s_{-i}).$$

Rearranging,

$$u_i(s'_i, s'_{-i}) - u_i(s_i, s'_{-i}) \geq u(s'_i, s_{-i}) - u_i(s_i, s_{-i}).$$

A supermodular game

A supermodular game is such that, for each $i \in N$,

S_i is sub-lattice in $\mathbb{R}^\ell$, u_i has increasing differences in (s_i, s_{-i}), and u_i is supermodular in s

$$u_i(s) + u_i(s') \leq u_i(s \wedge s') + u_i(s \vee s') \text{ for all } s, s' \in S.$$

Remark III: Supermodularity in s_i is implied by supermodularity in s (let $s = (s_i, s_{-i}), s' = (s'_i, s'_{-i})$, and $s_{-i} = s'_{-i}$).

Next, we give conditions for supermodularity in terms of derivatives of the payoff function u_i:

– (Topkis) If $S_i = \mathbb{R}^\ell$ and u_i is C^2 in s_i, then u_i is supermodular in s_i if and only if for each s_{-i}

$$\frac{\partial^2 u_i}{\partial s_{ik} \partial s_{ij}}(s_i, s_{-i}) \geq 0 \text{ for all } k, j = 1, ..., \ell.$$

– If $S = \mathbb{R}^{\ell n}$ and u_i is C^2, then u_i is supermodular if and only if

$$\frac{\partial^2 u_i}{\partial s_k \partial s_j}(s) \geq 0 \text{ for all } k, j = 1, ..., \ell n.$$

Proof: Let $e_k = (0, ..., 0, 1, 0, ..., 0)$ be an ℓn-vector with the unit in the k-th place, and let $u = (s + \varepsilon e_k)$ and $v = (s + \eta e_j)$ for $k \neq j$ and $\varepsilon, \eta > 0$. Supermodularity of u_i implies that

$$u_i(u) + u(v) \leq u_i(u \vee v) + u_i(u \wedge v).$$

Substituting ,

$$u_i(s + \varepsilon e_k) + u(s + \eta e_j) \leq u_i(s + \varepsilon e_k + \eta e_j) + u_i(s)$$

which implies that

$$\varepsilon\eta \frac{\partial^2 u_i}{\partial s_k \partial s_j}(s) \geq 0.$$

as required.

Examples

Cournot game: suppose $N = 1, 2$, $q_i = [0, \bar{q}_i]$, and $u_i(q_i, q_j) = q_i P_i(q_i, q_j) - C_i(q_i)$ where the inverse demand functions $P_i(q_i, q_j)$ are C^2, $P_i + q_i \partial P_i / \partial q_i$ is decreasing in q_i, and $C_i(q_i)$ is differentiable.

If $s_1 \equiv q_1$ and $s_2 \equiv -q_2$ then $\partial^2 u_i / \partial s_i \partial s_j \geq 0$ for all $i \neq j$. Thus, the game is supermodular.

Note: an increase in the strategy of player 2 reduces his output and this encourages player 1 to increase his output and his strategy.

Bertrand game: consider an oligopoly with demand functions

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j$$

where $b_i > 0$ and $d_{ij} > 0$.

Let $u_i(p_i, p_{-i}) = (p_i - c_i) D_i(p_i, p_{-i})$.

Then,

$$\frac{\partial^2 u_i}{\partial p_i \partial s_j} \geq 0$$

for all $i, j \neq i$.

Search: consider a matching technology $p(e, e^*) = ee^*$ – the probability of being matched with another player when the player being matched takes effort $e \in [0, 1]$ and the average effort of the other players is e^*. The cost of effort is $c(e) = e^2/2$.

The strategy set $[0, 1]$ is a sub-lattice of $\mathbb{R}$ and the payoff function $u(e, e^*) = ee^* - e^2/2$ has increasing first differences:

$$u(e, e^*) - u(\tilde{e}, e^*) = (e - \tilde{e})e^* - e^2/2 + \tilde{e}^2/2$$

is increasing in e^* when $e > \tilde{e}$. Because the strategy e is a scalar, the payoff function u is automatically supermodular in e.

Bank run: let $s_i = 0$ (resp. $s_i = 1$) represents a decision of player i to withdraw (resp. delay withdrawal). The payoff function $u_i(s)$ can be written

$$u_i(s) = (1 - s_i) + s_i R(\alpha(s)),$$

where

$$R(\alpha(s)) = \begin{cases} r > 1 & \text{if } \alpha(s) \leq \bar{\alpha} \\ 0 & \text{if } \alpha(s) > \bar{\alpha} \end{cases}$$

and $\alpha(s) = \sum_{i \in N} (1 - s_i)/n$ is the proportion of players who withdraw.

The set of strategy profiles is $S = \{0, 1\}^n$, which is easily seen to be a sub-lattice of $\mathbb{R}^n$.

Supermodularity of u_i in s_i follows automatically because s_i is one-dimensional. Also, $s_i > \tilde{s}_i$ implies $s_i = 1$ and $\tilde{s}_i = 0$, so

$$u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) = R(\alpha(s_i, s_{-i})) - 1.$$

Clearly, $R(\alpha(s_i, s_{-i}))$ is non-increasing in $\alpha(s_i, s_{-i})$ and $\alpha(s_i, s_{-i})$ is decreasing in s_{-i}, so u_i exhibits increasing first differences.

Applications of supermodularity

Supermodular games derive their interest from the following result (Tarski, 1951):

If S is non-empty, compact sub-lattice of $\mathbb{R}^\ell$ and $f : S \longrightarrow S$ is such that $f(x) \leq f(y)$ if $x \leq y$, then f has a fixed point s.t. $x^* = f(x^*)$ (i.e. f cannot "jump down").

Tarski's theorem is relevant since the set $BR(s_{-i})$ is a non-empty, compact sub-lattice and increases in s_{-i}.

$u_i : S \to \mathbb{R}$ is upper semi-continuous (u.s.c.) in s_i if

$$\limsup_q u_i(s_i^q, s_{-i}) \leq u_i(s_i^0, s_{-i}),$$

for any $s_{-i} \in S_i$ and any sequence $\{s_i^q\}$ in S_i such that $\lim_q s_i^q = s_i^0$.

Intuitively, $u_i(s_i, s_{-i})$ can jump up as s_i changes, but cannot jump down (a maintained assumption).

Result I: $BR_i(s_{-i})$ *is non-empty and compact for every* $s_{-i} \in S_{-i}$.

– Pick $s_{-i} \in S_{-i}$ and consider a sequence $\{s_i^q\}$ in S_i such that

$$\lim_q u_i(s_i^q, s_{-i}) = \sup\{u_i(s_i, s_{-i}) | s_i \in S_i\}.$$

Since S_i is compact, the sequence $\{s_i^q\}$ has a convergent subsequence with limit s_i^0 and WLOG we can use the same notation to denote the subsequence. Then u.s.c. implies that

$$\sup\{u_i(s_i, s_{-i}) | s_i \in S_i\} = \lim_q u_i(s_i^q, s_{-i}) \leq u_i(s_i^0, s_{-i}) < \infty.$$

Thus, $s_i^0 \in BR_i(s_{-i})$ as required.

- Suppose that $\{s_i^q\}$ is a sequence in $BR_i(s_{-i})$ for some fixed $s_{-i} \in S_{-i}$. Since S_i is compact, $\{s_i^q\}$ has a convergent subsequence with a limit $s_i^0 \in S_i$.

- WLOG, take $\{s_i^q\}$ to be the convergent subsequence. The u.s.c. of u_i in s_i implies that

$$\lim_q u_i(s_i^q, s_{-i}) \leq u_i(s_i^0, s_{-i}),$$

  so $s_i^0 \in BR_i(s_{-i})$.

- Thus, $BR_i(s_{-i})$ is closed and $BR_i(s_{-i}) \subset S_i$ shows that it is bounded, so $BR_i(s_{-i})$ is compact as claimed.

Result II: $BR_i(s_{-i})$ *is a sub-lattice of* S_i *for any* $s_{-i} \in S_{-i}$.

- The proof is by contradiction. Suppose that $s_i, s_i' \in BR_i(s_{-i})$ for some $s_{-i} \in S_{-i}$ and that $s_i \wedge s_i' \notin BR_i(s_{-i})$, that is

$$u_i(s_i \wedge s_i', s_{-i}) < u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}).$$

- Supermodularity in s_i implies that

$$u_i(s_i \vee s_i', s_{-i}) + u_i(s_i \wedge s_i', s_{-i}) \geq u_i(s_i, s_{-i}) + u_i(s_i', s_{-i}).$$

The two inequalities together imply that

$$u_i(s_i \wedge s_i', s_{-i}) > u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}),$$

contradicting the assumption that s_i and s_i' are best responses (the proof that $s_i \vee s_i' \in BR_i(s_{-i})$ is similar).

<u>Result III</u>: *(i) for every $s_{-i} \in S_i$, $BR_i(s_{-i})$ has a greatest element $\overline{BR}_i(s_{-i})$ (by Zorn's Lemma), and (ii) $\overline{BR}_i$ is monotonically non-decreasing, that is, for any $s_{-i}, s'_{-i} \in S_{-i}$,*

$$s_{-i} \leq s'_{-i} \Longrightarrow \overline{BR}_i(s_{-i}) \leq \overline{BR}_i(s'_{-i}).$$

– Suppose that $s_{-i}, s'_{-i} \in S_{-i}$, $s_{-i} \leq s'_{-i}$, $s_i \in \overline{BR}_i(s_{-i})$ and $s'_i \in \overline{BR}_i(s'_{-i})$. Supermodularity in s_i implies that

$$u_i(s_i \vee s'_i, s'_{-i}) + u_i(s_i \wedge s'_i, s'_{-i}) \geq u_i(s_i, s'_{-i}) + u_i(s'_i, s'_{-i}),$$

and thus

$$u_i(s_i \vee s'_i, s'_{-i}) - u_i(s'_i, s'_{-i}) \geq u_i(s_i, s'_{-i}) - u_i(s_i \wedge s'_i, s'_{-i}).$$

- Since $s_i \in \overline{BR}_i(s_{-i})$, increasing first differences implies that

$$u_i(s_i \vee s_i', s_{-i}') - u_i(s_i', s_{-i}') \geq u_i(s_i, s_{-i}) - u_i(s_i \wedge s_i', s_{-i}) \geq 0$$

- But, since $s_i' \in \overline{BR}_i(s_{-i}')$,

$$u_i(s_i \vee s_i', s_{-i}') - u_i(s_i', s_{-i}') = 0,$$

  so $s_i \vee s_i' \in \overline{BR}_i(s_{-i}')$.

- If s_i' is the largest element in $BR_i(s_{-i}')$ then $s_i' \geq s_i \vee s_i'$, which implies that $s_i' \geq s_i$.

Lattice properties of the fixed point set

Result IV: *the function* $\overline{BR}(\cdot) \equiv \overline{BR}_1(\cdot) \times ... \times \overline{BR}_n(\cdot)$ *mapping* S *into* S *has a fixed point.*

Result V (Topkis 1979): *if the game is supermodular and, for each player* i, S_i *is compact and* u_i *is u.s.c. in* s_i *for each* s_{-i}, *then the set of pure-strategy Nash equilibria is non-empty and contains greatest and least elements,* $\bar{s}$ *and* $\underline{s}$, *respectively.*

Result VI (Vives 1990): *if the game is strictly supermodular and, for each player i, S_i is compact and u_i is u.s.c. in s_i for each s_{-i}, then the set of pure strategy Nash equilibria is a non-empty, complete sub-lattice.*

(a sub-lattice is complete if the **sup** $\vee$ and **inf** $\wedge$ of *every* subset belongs to the sub-lattice).

Concluding, supermodular are <u>well-behaved</u>:

– pure-strategy equilibria,

– upper (and lower) bound of each player's equilibrium strategies.

– the upper and lower bounds of the sets of Nash equilibria and rationalizable strategies coincide (Milgrom and Roberts, 1990).

Repeated games

A repeated game is not supermodular even if the stage game on which it is based is supermodular. For example, consider the 2×2 coordination game:

	b_1	b_2
a_1	$1, 1$	$0, 0$
a_2	$0, 0$	$4, 4$

If we adopt the convention that $a_1 < a_2$ and $b_1 < b_2$ then this is a game with increasing first differences (and hence a supermodular game).

Suppose this game is played $T+1$ periods, and the payoff from the repeated game is simply the undiscounted sum of the payoffs in each of the stage games.

Consider the following strategy-pair: (a_1, b_2) in the first stage game, and in each subsequent game (b_1, b_2) if (a_1, b_2) is the outcome in the first stage and (a_1, b_1) otherwise. This pair of strategies constitutes a subgame perfect equilibrium of the repeated game.

Now consider the payoffs for player 1 that result from different outcomes in the first stage game:

Outcome	Payoff
(a_1, b_1)	$1 + T$
(a_2, b_1)	$0 + T$
(a_1, b_2)	$0 + 4T$
(a_2, b_2)	$4 + T$

The gain to player 1 from increasing his action from a_1 to a_2 is -1 when player 2 chooses b_1 and $4 - 3T$ when player 2 chooses b_2. Thus, an increase in player 2's action <u>reduces</u> the first difference of player 1 for T sufficiently large.