Appendix VIII
The generalized kinked specification

We continue to assume that state 2 has an objectively known probability \( \pi_2 = \frac{1}{3} \), whereas states 1 and 3 occur with unknown probabilities \( \pi_1 \) and \( \pi_3 \). The utility of a portfolio \( x = (x_1, x_2, x_3) \) takes the following form:

I. \( x_2 \leq x_{\min} \)
   \[ \alpha_1^1 u(x_2) + \alpha_2^1 u(x_{\min}) + \alpha_3^1 u(x_{\max}) \]

II. \( x_{\min} \leq x_2 \leq x_{\max} \)
   \[ \alpha_1^2 u(x_{\min}) + \alpha_2^2 u(x_2) + \alpha_3^2 u(x_{\max}) \]

III. \( x_{\max} \leq x_2 \)
    \[ \alpha_1^3 u(x_{\min}) + \alpha_2^3 u(x_{\max}) + \alpha_3^3 u(x_2) \]

where \( x_{\min} = \min\{x_1, x_3\} \) and \( x_{\max} = \max\{x_1, x_3\} \). This formulation (equation 3) embeds the kinked specification (equation 1) as a special case. At the end of this note, we show that, through a suitable change of variables, the generalized kinked specification can also be interpreted as reflecting Recursive Nonexpected Utility (RNEU) where the ambiguity is modeled as an equal probability that \( \pi_1 = \frac{2}{3} \) or \( \pi_3 = \frac{2}{3} \). We begin by deriving the optimality conditions.

[1] Parameter restrictions

[1.1] Consistency
When \( x_2 = x_{\min} \), consistency requires that
\[
(\alpha_1^1 + \alpha_2^1) u(x_{\min}) + \alpha_3^1 u(x_{\max}) = (\alpha_1^2 + \alpha_2^2) u(x_{\min}) + \alpha_3^2 u(x_{\max}).
\]
Without loss of generality we can assume that
\[
\alpha_1^1 + \alpha_2^1 + \alpha_3^1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2,
\]
in which case the equation preceding the last implies that
\[
(\alpha_1^1 + \alpha_2^1) [u(x_{\min}) - u(x_{\max})] = (\alpha_1^2 + \alpha_2^2) [u(x_{\min}) - u(x_{\max})]
\]
or
\[
\alpha_1^1 + \alpha_2^1 = \alpha_1^2 + \alpha_2^2.
\]
Similarly, when \( x_2 = x_{\max} \) consistency requires that
\[
\alpha_2^2 + \alpha_3^2 = \alpha_2^3 + \alpha_3^3.
\]
We further normalize the coefficients so that
\[
\alpha_1^j + \alpha_2^j + \alpha_3^j = 1 \text{ for all } j.
\]
This leads to the following:
\[ \alpha_3^1 = \alpha_3^2, \alpha_1^2 = \alpha_1^3. \]

[1.2] Reparametrization

Let
\[ \alpha_1^1 = \beta_1, \ \alpha_1^1 + \alpha_2^1 = \beta_2, \]
\[ \alpha_1^2 = \beta_3, \ \alpha_1^3 + \alpha_2^2 = \beta_4. \]

Using the consistency conditions, the original coefficients are reparametrized as follows:
\[ \alpha_1^1 = \beta_1, \ \alpha_2^1 = \beta_2 - \beta_1, \ \alpha_3^1 = 1 - \beta_2, \]
\[ \alpha_1^2 = \beta_3, \ \alpha_2^2 = \beta_2 - \beta_3, \ \alpha_3^2 = 1 - \beta_2, \]
\[ \alpha_1^3 = \beta_3, \ \alpha_2^3 = \beta_4 - \beta_3, \ \alpha_3^3 = 1 - \beta_4. \]

Note that \( \beta_1 \leq \beta_2 \leq 1, \beta_3 \leq \beta_2 \) and \( \beta_3 \leq \beta_4 \). The utility of a portfolio \( x = (x_1, x_2, x_3) \) can be written with parameters \( \beta_1, ..., \beta_4 \):

I. \( x_2 \leq x_{\text{min}} \)
\[ \beta_1 u(x_2) + (\beta_2 - \beta_1) u(x_{\text{min}}) + (1 - \beta_2) u(x_{\text{max}}) \]

II. \( x_{\text{min}} \leq x_2 \leq x_{\text{max}} \)
\[ \beta_3 u(x_{\text{min}}) + (\beta_2 - \beta_3) u(x_2) + (1 - \beta_2) u(x_{\text{max}}) \]

III. \( x_{\text{max}} \leq x_2 \)
\[ \beta_3 u(x_{\text{min}}) + (\beta_4 - \beta_3) u(x_{\text{max}}) + (1 - \beta_4) u(x_2) \]

We adopt a simpler three-parameter model, in which the parameter \( \delta \) measures the ambiguity attitudes, the parameter \( \gamma \) measures pessimism/optimism, and \( \rho \) is the coefficient of absolute risk aversion. The mapping from the two parameters \( \delta \) and \( \gamma \) to the four parameters \( \beta_1, ..., \beta_4 \) is given by the equations
\[ \beta_1 = \frac{1}{3} + \gamma \]
\[ \beta_2 = \frac{2}{3} + \gamma + \delta \]
\[ \beta_3 = \frac{1}{3} + \gamma + \delta \]
\[ \beta_4 = \frac{2}{3} + \gamma, \]

with \( -\frac{1}{3} < \delta, \gamma < \frac{1}{3} \) and \( -\frac{1}{3} < \delta + \gamma < \frac{1}{3} \) so that the decision weight attached to each payoff in equation 3 is nonnegative.

By the symmetry property between $x_1$ and $x_3$, we know that $x_1 \leq x_3$ if and only if $p_1 \geq p_3$. We can use this fact to identify the price of $x_{\text{min}}$ as $p_{\text{max}} = \max\{p_1, p_3\}$. Similarly, we can identify the price of $x_{\text{max}}$ as $p_{\text{min}} = \min\{p_1, p_3\}$. For the rest of the note, we denote

$$x_i = x_{\text{min}} \text{ and } x_j = x_{\text{max}},$$

$$p_i = p_{\text{max}} \text{ and } p_j = p_{\text{min}}.$$

The maximization of the generalized kinked utility function can be broken down into three sub-problems:

- **SP1:** $x_2 \leq x_i$
  
  $$\max_x \left( \frac{1}{3} + \gamma \right) u(x_2) + \left( \frac{1}{3} + \delta \right) u(x_i) + \left( \frac{1}{3} - \gamma - \delta \right) u(x_j)$$

  subject to $p \cdot x = 1$, $x_j - x_i \geq 0$ and $x_i - x_2 \geq 0$.

- **SP2:** $x_i \leq x_2 \leq x_j$
  
  $$\max_x \left( \frac{1}{3} + \gamma + \delta \right) u(x_2) + \left( \frac{1}{3} \right) u(x_i) + \left( \frac{1}{3} - \gamma - \delta \right) u(x_j)$$

  subject to $p \cdot x = 1$, $x_j - x_2 \geq 0$ and $x_2 - x_i \geq 0$.

- **SP3:** $x_j \leq x_2$
  
  $$\max_x \left( \frac{1}{3} + \gamma + \delta \right) u(x_j) + \left( \frac{1}{3} - \delta \right) u(x_i) + \left( \frac{1}{3} - \gamma \right) u(x_2)$$

  subject to $p \cdot x = 1$, $x_j - x_i \geq 0$, and $x_2 - x_j \geq 0$.

We adopt the CARA utility function $u(x) = -\frac{1}{\rho} e^{-\rho x}$. Instead of characterizing the exact conditions of prices and model parameters that tell which sub-problem the optimal solution of demands belongs to, we can adopt the following two-step algorithm computing a (globally) optimal demand:

**Step 1** Given a price vector $p$ and parameter values $(\delta, \gamma, \rho)$, compute a (locally) optimal solution in each of the three sub-problems.

**Step 2** Compare the utilities of locally optimal solutions of three sub-problems and choose one yielding the highest utility as a (globally) optimal solution of demand.

In what follows, we characterize optimal demand with conditions on parameters in each sub-problem. Due to the fact that the CARA utility function generates a boundary solution for certain price vectors, we first set up the Lagrangian function for optimal solutions without the non-negativity condition of demand and impose that condition later, for computational ease.
[2.1] SP1: $x_2 \leq x_i$

The Lagrangian function without the non-negativity condition of demand is given by

$$L(x) = \left(\frac{1}{3} + \gamma\right) u(x_2) + \left(\frac{1}{3} + \delta\right) u(x_i) + \left(\frac{1}{3} - \gamma - \delta\right) u(x_j) + \lambda_1 (x_i - x_2) + \lambda_2 (x_j - x_i) + \mu (1 - p_1 x_1 - p_2 x_2 - p_3 x_3).$$

The necessary conditions for the maximization problem are given by

$$L_2(x) = \left(\frac{1}{3} + \gamma\right) \exp(-\rho x_2) - \lambda_1 - \mu p_2 = 0,$$
$$L_i(x) = \left(\frac{1}{3} + \delta\right) \exp(-\rho x_i) + \lambda_1 - \lambda_2 - \mu p_i = 0,$$
$$L_j(x) = \left(\frac{1}{3} - \gamma - \delta\right) \exp(-\rho x_j) + \lambda_2 - \mu p_j = 0,$$
$$\lambda_1 (x_i - x_2) = 0 = \lambda_2 (x_j - x_i), \lambda_1 \geq 0, \lambda_2 \geq 0,$$
$$x_i - x_2 \geq 0, x_j - x_i \geq 0,$$
$$1 = p_1 x_1 + p_2 x_2 + p_3 x_3, \mu > 0.$$

[2.1.1] $\lambda_1 > 0$ and $\lambda_2 > 0$

This implies that $x_1^* = x_2^* = x_j^*$. Then the optimal demand is given by

$$x_1^* = x_2^* = x_3^* = \frac{1}{p_1 + p_2 + p_3}.$$

For the parameter conditions leading to this solution, we need to check the following:

$$\left(\frac{1}{3} + \gamma\right) \exp(-\rho x_2) > \mu p_2,$$
$$\left(\frac{1}{3} - \gamma - \delta\right) \exp(-\rho x_j) < \mu p_j,$$
$$\left(\frac{2}{3} + \gamma + \delta\right) \exp(-\rho x_i) > \mu (p_2 + p_i),$$
$$\left(\frac{2}{3} - \gamma\right) \exp(-\rho x_j) < \mu (p_1 + p_3),$$
which yields the following inequality conditions under the optimal solution:

$$\ln \left(\frac{p_2}{p_j}\right) < \ln \left(\frac{1}{3} + \gamma\right) \left(\frac{1}{3} - \gamma - \delta\right),$$
$$\ln \left(\frac{p_2}{p_1 + p_3}\right) < \ln \left(\frac{1}{3} + \gamma\right) \left(\frac{2}{3} - \gamma\right),$$
$$\ln \left(\frac{p_2 + p_i}{p_j}\right) < \ln \left(\frac{2}{3} + \gamma + \delta\right) \left(\frac{2}{3} - \gamma - \delta\right).$$
[2.1.2] $\lambda_1 = 0$ and $\lambda_2 > 0$

This implies that $x_1^* = x_3^* > x_2^*$. The solution without non-negativity condition is given by

$$x_2^* = \frac{1}{p_1 + p_2 + p_3} - \frac{(p_1 + p_3)}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_1 + p_3} \right) - \ln \left( \frac{1 + \gamma}{\frac{2}{3} - \gamma} \right) \right],$$

$$x_1^* = x_3^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_2}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_1 + p_3} \right) - \ln \left( \frac{1 + \gamma}{\frac{2}{3} - \gamma} \right) \right].$$

The inequality conditions for this solution are given by

$$\ln \left( \frac{p_2}{p_1 + p_3} \right) > \ln \left( \frac{1 + \gamma}{\frac{2}{3} - \gamma} \right),$$

$$\ln \left( \frac{p_1}{p_j} \right) < \ln \left( \frac{1 + \delta}{\frac{2}{3} - \gamma - \delta} \right).$$

If $x_2^* \geq 0$, then the optimal demand is

$$x_2^* = \frac{1}{p_1 + p_2 + p_3} - \frac{(p_1 + p_3)}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_1 + p_3} \right) - \ln \left( \frac{1 + \gamma}{\frac{2}{3} - \gamma} \right) \right],$$

$$x_1^* = x_3^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_2}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_1 + p_3} \right) - \ln \left( \frac{1 + \gamma}{\frac{2}{3} - \gamma} \right) \right].$$

If $x_2^* < 0$, then the optimal demand is given by

$$x_2^* = 0 \text{ and } x_1^* = x_3^* = \frac{1}{p_2 + p_3}.$$

[2.1.3] $\lambda_1 > 0$ and $\lambda_2 = 0$

This implies that $x_2^* = x_3^* < x_j^*$. The solution without non-negativity condition is given by

$$x_2^* = x_3^* = \frac{1}{p_1 + p_2 + p_3} - \frac{p_j}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2 + p_i}{p_j} \right) - \ln \left( \frac{2 + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right) \right],$$

$$x_j^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_2 + p_i}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2 + p_i}{p_j} \right) - \ln \left( \frac{2 + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right) \right].$$

The inequality condition for this solution is given by

$$\ln \left( \frac{p_2 + p_i}{p_j} \right) > \ln \left( \frac{2 + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right),$$

$$\ln \left( \frac{p_2}{p_i} \right) < \ln \left( \frac{1 + \gamma}{\frac{2}{3} + \delta} \right).$$
If \( x^*_2 = x^*_i \geq 0 \), the optimal demand will be the same as above:

\[
x^*_2 = x^*_i = \frac{1}{p_1 + p_2 + p_3} - \frac{p_j}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2 + p_i}{p_j} \right) - \ln \left( \frac{\frac{2}{3} + \gamma + \delta}{\frac{3}{3} - \gamma - \delta} \right) \right],
\]

\[
x^*_j = \frac{1}{p_1 + p_2 + p_3} + \frac{p_2 + p_i}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2 + p_i}{p_j} \right) - \ln \left( \frac{\frac{2}{3} + \gamma + \delta}{\frac{3}{3} - \gamma - \delta} \right) \right].
\]

If \( x^*_2 = x^*_i < 0 \), the optimal demand will be

\[
x^*_2 = x^*_i = 0 \text{ and } x^*_j = \frac{1}{p_j}.
\]

[2.1.4] \( \lambda_1 = 0 \) and \( \lambda_2 = 0 \)

This implies that \( x^*_j > x^*_i > x^*_2 \). The solution without non-negativity condition is given by

\[
x^*_2 = \frac{1}{p_1 + p_2 + p_3} - \frac{p_i}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_i} \right) - \ln \left( \frac{\frac{1}{3} + \gamma}{\frac{3}{3} - \gamma - \delta} \right) \right],
\]

\[
x^*_j = \frac{1}{p_1 + p_2 + p_3} + \frac{p_2 + p_j}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_j} \right) - \ln \left( \frac{\frac{1}{3} + \gamma}{\frac{3}{3} - \gamma - \delta} \right) \right],
\]

\[
x^*_j = \frac{1}{p_1 + p_2 + p_3} - \frac{p_j}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2 + p_i}{p_j} \right) - \ln \left( \frac{\frac{1}{3} + \gamma}{\frac{3}{3} - \gamma - \delta} \right) \right] + \frac{p_2 + p_i}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2 + p_i}{p_j} \right) - \ln \left( \frac{\frac{1}{3} + \gamma}{\frac{3}{3} - \gamma - \delta} \right) \right].
\]

If the non-negativity condition for each asset is satisfied, then the above solution is the optimal demand from the problem with the non-negativity condition of demands. Otherwise, we need to further refine the problem by setting an asset violating the non-negativity condition to be zero. There are two cases to consider: (i) \( x^*_2 < x^*_i < 0 \), (ii) \( x^*_2 < 0 \) and \( x^*_i > 0 \).

(i) \( x^*_2 < x^*_i < 0 \)

The optimal solution is then given by

\[
x^*_j = \frac{1}{p_j} \text{ and } x^*_2 = x^*_i = 0.
\]

(ii) \( x^*_2 < 0 \) and \( x^*_i > 0 \)
The solution to the problem by imposing that $x_2^* = 0$ is given by

$$
x_i' = \frac{1}{p_1 + p_3} - \frac{p_j}{\rho (p_1 + p_3)} \left[ \ln \left( \frac{p_i}{p_j} \right) - \ln \left( \frac{1/3 + \delta}{1/3 - \gamma - \delta} \right) \right],
$$

$$
x_j' = \frac{1}{p_1 + p_3} + \frac{p_i}{\rho (p_1 + p_3)} \left[ \ln \left( \frac{p_i}{p_j} \right) - \ln \left( \frac{1/3 + \delta}{1/3 - \gamma - \delta} \right) \right].
$$

If $x_i' \geq 0$, then the solution with $x_2^* = 0$ is the optimal one in the original problem with the non-negativity condition of demands:

$$
x_2^* = 0, x_i^* = x_i' \text{ and } x_j^* = x_j'.
$$

If $x_i' < 0$, then the optimal solution is given by

$$
x_2^* = x_i^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.
$$

[2.2] SP2: $x_i \leq x_2 \leq x_j$  

The Lagrangian function without the non-negativity condition of demand is given by

$$
\mathcal{L}(x) = \left( \frac{1}{3} + \gamma + \delta \right) u(x_i) + \left( \frac{1}{3} \right) u(x_2) + \left( \frac{1}{3} - \gamma - \delta \right) u(x_j) + \lambda_1 (x_j - x_2) + \lambda_2 (x_2 - x_i) + \mu \left( 1 - p_1 x_1 - p_2 x_2 - p_3 x_3 \right).
$$

The necessary conditions for the maximization problem are given by

$$
\mathcal{L}_i(x) = \left( \frac{1}{3} + \gamma + \delta \right) \exp (-\rho x_i) - \lambda_2 - \mu p_i = 0,
$$

$$
\mathcal{L}_2(x) = \left( \frac{1}{3} \right) \exp (-\rho x_2) - \lambda_1 + \lambda_2 - \mu p_2 = 0,
$$

$$
\mathcal{L}_j(x) = \left( \frac{1}{3} - \gamma - \delta \right) \exp (-\rho x_j) + \lambda_1 - \mu p_j = 0,
$$

$$
0 = \lambda_2 (x_2 - x_i) = \lambda_1 (x_j - x_2), \lambda_1 \geq 0, \lambda_2 \geq 0, x_j - x_2 \geq 0, x_2 - x_i \geq 0, \mu > 0, 1 - p_1 x_1 - p_2 x_2 - p_3 x_3 = 0.
$$

[2.2.1] $\lambda_1 > 0$ and $\lambda_2 > 0$  

This implies that $x_i^* = x_2^* = x_j^*$. Thus, the optimal demand is given by

$$
x_1^* = x_2^* = x_3^* = \frac{1}{p_1 + p_2 + p_3}.
$$
We need to check the following parameter conditions for the optimal demand:

\[
\left(\frac{1}{3} + \gamma + \delta\right) \exp\{-\rho x_i\} > \mu p_i,
\]
\[
\left(\frac{1}{3} - \gamma - \delta\right) \exp\{-\rho x_j\} < \mu p_j,
\]
\[
\left(\frac{2}{3} + \gamma + \delta\right) \exp\{-\rho x_2\} > \mu (p_i + p_2),
\]
\[
\left(\frac{2}{3} - \gamma - \delta\right) \exp\{-\rho x_2\} < \mu (p_2 + p_j).
\]

Then we have the following inequality conditions for model parameters:

\[
\ln \left(\frac{p_i}{p_j}\right) < \ln \left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right),
\]
\[
\ln \left(\frac{p_i}{p_2 + p_j}\right) < \ln \left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right),
\]
\[
\ln \left(\frac{p_i + p_2}{p_2}\right) < \ln \left(\frac{\frac{2}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right).
\]

[2.2.2] \(\lambda_1 = 0\) and \(\lambda_2 > 0\)

This implies that \(x_2^* = x_i^* < x_j^*\). The optimal demand without the non-negativity condition is given by

\[
x_2^* = x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{p_j}{\rho (p_1 + p_2 + p_3)} \left[\ln \left(\frac{p_i + p_2}{p_j}\right) - \ln \left(\frac{\frac{2}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right)\right],
\]
\[
x_j^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_2 + p_i}{\rho (p_1 + p_2 + p_3)} \left[\ln \left(\frac{p_i + p_2}{p_j}\right) - \ln \left(\frac{\frac{2}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right)\right].
\]

The parameter condition for this solution is given by

\[
\ln \left(\frac{p_i + p_2}{p_j}\right) > \ln \left(\frac{\frac{2}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right),
\]
\[
\ln \left(\frac{p_i}{p_2}\right) < \ln \left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3}}\right).
\]

If \(x_2^* = x_i^* \geq 0\), then the above solution is the optimal one from the original maximization problem. Otherwise, the optimal solution with the non-negativity condition is given by

\[
x_2^* = x_i^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.
\]
[2.2.3] \( \lambda_1 > 0 \) and \( \lambda_2 = 0 \)

This implies that \( x_j^* = x_2^* > x_i^* \). The optimal demand without the non-negativity condition is given by

\[
x_j^* = x_2^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2 + p_j} \right) - \ln \left( \frac{1}{3} + \gamma + \delta \right) \right],
\]

\[
x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{p_j}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_j} \right) - \ln \left( \frac{1}{3} + \gamma + \delta \right) \right].
\]

The parameter condition for this solution is given by

\[
\ln \left( \frac{p_i}{p_2 + p_j} \right) > \ln \left( \frac{1}{3} + \gamma + \delta \right),
\]

\[
\ln \left( \frac{p_2}{p_j} \right) < \ln \left( \frac{1}{3} + \gamma + \delta \right).
\]

If \( x_i^* \geq 0 \), the optimal demand from the original problem will be the same as above. Otherwise, the optimal demand with the non-negativity condition is

\[
x_i^* = 0 \quad \text{and} \quad x_2^* = x_j^* = \frac{1}{p_2 + p_j}.
\]

[2.2.4] \( \lambda_1 = 0 \) and \( \lambda_2 = 0 \)

This implies that \( x_j^* > x_2^* > x_i^* \). The optimal solution without the non-negativity condition is given by

\[
x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{(p_2 + p_j)}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_j} \right) - \ln \left( \frac{1}{3} + \gamma + \delta \right) \right],
\]

\[
x_2^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2} \right) - \ln \left( \frac{1}{3} + \gamma + \delta \right) \right],
\]

\[
x_j^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2} \right) - \ln \left( \frac{1}{3} + \gamma + \delta \right) \right]
+ \frac{p_i}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_2}{p_j} \right) - \ln \left( \frac{1}{3} + \gamma + \delta \right) \right].
\]

If the non-negativity condition for each asset is satisfied, then the above solution is the optimal demand from the problem with the non-negativity condition of demands. Otherwise, we need to further refine the problem by setting an asset violating the non-negativity condition to be zero. There are two cases to consider: (i) \( x_i^* < x_2^* < 0 \), (ii) \( x_i^* < 0 \) and \( x_2^* > 0 \).
(i) $x_i^* < x_2^* < 0$

The optimal solution is then given by

$$x_i^* = x_2^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.$$ 

(ii) $x_i^* < 0 \text{ and } x_2^* > 0$

By imposing that $x_i^* = 0$, we have the new solution as

$$x_i' = \frac{1}{p_2 + p_j} - \frac{p_j}{\rho (p_2 + p_j)} \left[ \ln \left( \frac{p_2}{p_j} \right) - \ln \left( \frac{1}{3} \right) \right],$$

$$x_j' = \frac{1}{p_2 + p_j} + \frac{p_j}{\rho (p_2 + p_j)} \left[ \ln \left( \frac{p_2}{p_j} \right) - \ln \left( \frac{1}{3} \right) \right].$$

If $x_2' \geq 0$, then the optimal demand from the original problem will be

$$x_i^* = 0, x_2^* = x_2' \text{ and } x_j^* = x_j'.$$

If $x_2' < 0$, then the optimal demand will be

$$x_i^* = x_2^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.$$ 

[2.3] **SP3**: $x_j \leq x_2$

The Lagrangian function without the non-negativity condition is given by

$$L(x) = \left( \frac{1}{3} + \gamma + \delta \right) u(x_i) + \left( \frac{1}{3} - \delta \right) u(x_j) + \left( \frac{1}{3} - \gamma \right) u(x_2) + \lambda_1 (x_2 - x_j) + \lambda_2 (x_j - x_i) + \mu (1 - p_1 x_1 - p_2 x_2 - p_3 x_3).$$

The necessary conditions for the maximization problem are given by

$$L_i(x) = \left( \frac{1}{3} + \gamma + \delta \right) \exp(-\rho x_i) - \lambda_2 - \mu p_i = 0,$$

$$L_j(x) = \left( \frac{1}{3} - \delta \right) \exp(-\rho x_j) - \lambda_1 + \lambda_2 - \mu p_j = 0,$$

$$L_2(x) = \left( \frac{1}{3} - \gamma \right) \exp(-\rho x_2) + \lambda_1 - \mu p_2 = 0,$$

$$0 = \lambda_1 (x_2 - x_j) = \lambda_2 (x_j - x_i), \lambda_1, \lambda_2 \geq 0,$$

$$\mu > 0 \text{ and } 1 - p_1 x_1 - p_2 x_2 - p_3 x_3 = 0.$$
\[2.3.1\] \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \)
This implies that \( x_2^* = x_j^* = x_i^* \). The optimal solution from the original problem is then given by
\[
x_2^* = x_3^* = \frac{1}{p_1 + p_2 + p_3},
\]
The parameter conditions for this solution are given by
\[
\ln \left( \frac{p_i}{p_2} \right) < \ln \left( \frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3} - \gamma} \right),
\]
\[
\ln \left( \frac{p_i}{p_2 + p_j} \right) < \ln \left( \frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3} - \gamma - \delta} \right),
\]
\[
\ln \left( \frac{p_1 + p_3}{p_2} \right) < \ln \left( \frac{\frac{2}{3} + \gamma}{\frac{2}{3} - \gamma} \right).
\]

\[2.3.2\] \( \lambda_1 = 0 \) and \( \lambda_2 > 0 \)
This implies that \( x_j^* = x_i^* < x_2^* \). The optimal solution without the non-negativity condition is given by
\[
x_2^* = \frac{1}{p_1 + p_2 + p_3} + \frac{(p_1 + p_3)}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_1 + p_3}{p_2} \right) - \ln \left( \frac{\frac{2}{3} + \gamma}{\frac{2}{3} - \gamma} \right) \right],
\]
The parameter conditions for this solution are given by
\[
\ln \left( \frac{p_1 + p_3}{p_2} \right) > \ln \left( \frac{\frac{2}{3} + \gamma}{\frac{2}{3} - \gamma} \right),
\]
\[
\ln \left( \frac{p_i}{p_j} \right) < \ln \left( \frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3} - \delta} \right),
\]
If \( x_1^* = x_3^* \geq 0 \), then the optimal solution from the original problem is the same as above. Otherwise, the optimal demand with the non-negativity condition is given by
\[
x_2^* = x_3^* = 0 \text{ and } x_2^* = \frac{1}{p_2}.
\]

\[2.3.3\] \( \lambda_1 > 0 \) and \( \lambda_2 = 0 \)
This implies that \( x_2^* = x_j^* > x_i^* \). The optimal demand without the non-negativity condition is given by
\[
x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{(p_2 + p_1)}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2 + p_j} \right) - \ln \left( \frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right) \right],
\]
\[
x_2^* = x_j^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho (p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2 + p_j} \right) - \ln \left( \frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right) \right].
\]
The parameter condition for this solution is given by

\[
\ln \left( \frac{p_i}{p_2 + p_j} \right) > \ln \left( \frac{1 + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right),
\]

\[
\ln \left( \frac{p_j}{p_2} \right) < \ln \left( \frac{1 - \delta}{\frac{2}{3} - \gamma} \right).
\]

If \( x_i^* \geq 0 \), then the optimal demand from the original problem is the same as above. Otherwise, the optimal demand with the non-negativity condition is given by

\[
x_i^* = 0 \quad \text{and} \quad x_j^* = \frac{1}{p_2 + p_j}.
\]

[2.3.4] \( \lambda_1 = 0 \) and \( \lambda_2 = 0 \)

The conditions imply that \( x_2^* > x_j^* > x_i^* \). The optimal demand without the non-negativity condition is given by

\[
x_2^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2} \right) - \ln \left( \frac{1 + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right) \right]
\]

\[
+ \frac{p_j}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_j}{p_2} \right) - \ln \left( \frac{1 - \delta}{\frac{2}{3} - \gamma} \right) \right],
\]

\[
x_j^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2} \right) - \ln \left( \frac{1 + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right) \right]
\]

\[
- \frac{(p_2 + p_j)}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_j}{p_2} \right) - \ln \left( \frac{1 - \delta}{\frac{2}{3} - \gamma} \right) \right],
\]

\[
x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{(p_2 + p_j)}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_i}{p_2} \right) - \ln \left( \frac{1 + \gamma + \delta}{\frac{2}{3} - \gamma - \delta} \right) \right]
\]

\[
+ \frac{p_j}{\rho(p_1 + p_2 + p_3)} \left[ \ln \left( \frac{p_j}{p_2} \right) - \ln \left( \frac{1 - \delta}{\frac{2}{3} - \gamma} \right) \right].
\]

If the non-negativity condition for each asset is satisfied, then the above solution is the optimal demand from the problem with the non-negativity condition of demands. Otherwise, we need to further refine the problem by setting an asset violating the non-negativity condition to be zero. There are two cases to consider: (i) \( x_i^* < x_j^* < 0 \), (ii) \( x_i^* < 0 \) and \( x_j^* > 0 \).

(i) \( x_i^* < x_j^* < 0 \)

Then the optimal solution from the original problem is given by

\[
x_1^* = x_3^* = 0 \quad \text{and} \quad x_2^* = \frac{1}{p_2},
\]
(ii) $x_i^* < 0$ and $x_j^* > 0$

By imposing that $x_i^* = 0$, we have the following new solution as

$$x_2^* = \frac{1}{p_2 + p_j} + \frac{p_j}{\rho \, (p_2 + p_j)} \left[ \ln \left( \frac{p_j}{p_2} \right) - \ln \left( \frac{\frac{1}{3} - \delta}{\frac{1}{3} - \gamma} \right) \right],$$

$$x_j^* = \frac{1}{p_2 + p_j} - \frac{p_2}{\rho \, (p_2 + p_j)} \left[ \ln \left( \frac{p_j}{p_2} \right) - \ln \left( \frac{\frac{1}{3} - \delta}{\frac{1}{3} - \gamma} \right) \right].$$

If $x_j' \geq 0$, then the optimal demand from the original problem is given by

$$x_i^* = 0, x_j^* = x_j' \text{ and } x_2^* = x_2'.$$

If $x_j' < 0$, then the optimal demand from the original problem is given by

$$x_i^* = x_j^* = 0 \text{ and } x_2^* = \frac{1}{p_2}.$$

[2.4] Non-uniqueness of the optimal demand

Finally we note that when $\delta < 0$ and/or $\gamma < 0$, the optimal demand is not unique when $p_k = p_{k'}$ for some $k \neq k' = 1, 2, 3$ because the generalized kinked utility function is not quasi-convex everywhere. Nevertheless, the utility function is not quasi-convex in each sub-problem. The above characterization of the optimal demands incorporates the cases of non-uniqueness.

Finally, we show that the generalized kinked specification can also be interpreted as reflecting a special case of RNEU where there is an equal probability that \( \pi_1 = \frac{2}{3} \) or \( \pi_3 = \frac{2}{3} \). Consider the following two-stage recursive Rank-Dependent Utility (RDU) model. Given a fixed underlying distribution \( \pi = (\pi_1, \pi_2, \pi_3) \), the first-stage rank-dependent expected utility \( V_{\pi} \) is given by

\[
V_{\left(\frac{2}{3}, \frac{1}{3}, 0\right)}(x) = [1 - w\left(\frac{1}{3}\right)] \max \{u(x_1), u(x_2)\} + w\left(\frac{1}{3}\right) \min \{u(x_1), u(x_2)\},
\]

\[
V_{\left(0, \frac{1}{3}, \frac{2}{3}\right)}(x) = [1 - w\left(\frac{1}{3}\right)] \max \{u(x_2), u(x_3)\} + w\left(\frac{1}{3}\right) \min \{u(x_2), u(x_3)\}.
\]

The second stage takes the rank-dependent expectation of the first-stage rank-dependent expected utilities:

\[
U(x) = [1 - w\left(\frac{1}{3}\right)] \max \left\{V_{\left(\frac{2}{3}, \frac{1}{3}, 0\right)}(x), V_{\left(0, \frac{1}{3}, \frac{2}{3}\right)}(x)\right\}
\]

\[
+ w\left(\frac{1}{3}\right) \min \left\{V_{\left(\frac{2}{3}, \frac{1}{3}, 0\right)}(x), V_{\left(0, \frac{1}{3}, \frac{2}{3}\right)}(x)\right\},
\]

and the decision weights can be expressed as follows:

\[
\beta_1 = w\left(\frac{1}{3}\right),
\]

\[
\beta_2 - \beta_1 = w\left(\frac{1}{3}\right) [1 - w\left(\frac{1}{3}\right)],
\]

\[
\beta_3 = w\left(\frac{1}{3}\right) w\left(\frac{1}{3}\right),
\]

\[
\beta_4 - \beta_3 = [1 - w\left(\frac{1}{3}\right)] w\left(\frac{1}{3}\right).
\]

Now consider the three relevant cases:

I. \( x_2 \leq x_{\min} \)

\[
U(x) = [1 - w\left(\frac{1}{3}\right)] \{[1 - w\left(\frac{1}{3}\right)] u(x_{\max}) + w\left(\frac{1}{3}\right) u(x_2)\}
\]

\[
+ w\left(\frac{1}{3}\right) \{[1 - w\left(\frac{1}{3}\right)] u(x_{\min}) + w\left(\frac{1}{3}\right) u(x_2)\}.
\]

Rearranging,

\[
U(x) = \beta_1 u(x_2) + (\beta_2 - \beta_3) u(x_{\min}) + (1 - \beta_2) u(x_{\max}).
\]

II. \( x_{\min} \leq x_2 \leq x_{\max} \)

\[
U(x) = [1 - w\left(\frac{1}{3}\right)] \{[1 - w\left(\frac{1}{3}\right)] u(x_{\max}) + w\left(\frac{1}{3}\right) u(x_2)\}
\]

\[
+ w\left(\frac{1}{3}\right) \{[1 - w\left(\frac{1}{3}\right)] u(x_2) + w\left(\frac{1}{3}\right) u(x_{\min})\}.
\]

Rearranging,

\[
U(x) = \beta_3 u(x_{\min}) + (\beta_2 - \beta_3) u(x_2) + (1 - \beta_2) u(x_{\max}).
\]

III. \( x_{\max} \leq x_2 \)

\[
U(x) = [1 - w\left(\frac{1}{3}\right)] \{[1 - w\left(\frac{1}{3}\right)] u(x_2) + w\left(\frac{1}{3}\right) u(x_{\max})\}
\]

\[
+ w\left(\frac{1}{3}\right) \{[1 - w\left(\frac{1}{3}\right)] u(x_2) + w\left(\frac{1}{3}\right) u(x_{\min})\}.
\]

Rearranging,

\[
U(x) = \beta_3 u(x_{\min}) + (\beta_4 - \beta_3) u(x_{\max}) + (1 - \beta_4) u(x_2).
\]