Financial Networks

By Douglas M. Gale and Shachar Kariv[1]

Networks are natural tools for understanding complex social and economic phenomena. Examples are: technology diffusion; neighborhood effects; financial crises and contagion; social learning; globalization. The surveys by Sanjeev Goyal (2003) and Matthew O. Jackson (2003) provide many examples. We can also think of markets as being structured like networks. The traditional Walrasian theory assumes that commodities are traded on a centralized exchange. The closest counterparts in the real world are stock exchanges, like the NYSE or Nasdaq, or commodity exchanges, like the CBOT. Decentralized trade, on the other hand, is naturally thought of as taking the form of a network. For example, the concept of a network can be to model the market for interbank lending and study the phenomenon of contagion. Applications of this idea can be found in Franklin Allen and Douglas M. Gale (2000); Xavier Freixas, Bruno M. Parigi and Jean-Charles Rochet (2000); Craig H. Furfine (2003); Michael Boss, Helmut Elsinger, Martin Summer and Stefan Thurner (2004); Hans A. Degryse and Gregory Nguyen (2004); and Christian Upper and Andreas Worms (2004). Recent theoretical research has begun to recognize that financial markets contain interesting and important frictions. For example, Darrell Duffie, Nicolae B. Garleanu and Lasse H. Pedersen (2005) model over-the-counter markets using a dynamic matching and bargaining framework. In their model, traders search randomly for trading opportunities. When a buyer and seller are matched, they bargain over the terms of trade. Search is time-consuming and hence costly, but in the limit as the length of the time period becomes vanishingly small, trade occurs instantaneously, the market becomes "frictionless" and the allocation of assets becomes approximately efficient.

A network structure provides an alternative model of decentralized financial exchange. In place of a continuum of agents and random matching, we assume a finite number of traders organized in a network that is represented by a *directed* graph. Each trader represents a node of the graph and a directed edge from trader i to trader j represents the possibility of trade between i and j. If the network is *complete*, that is, every possible trading opportunity is

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present, the situation is very similar to the centralized auction market. What makes the network model interesting is the assumption of incompleteness. A network is *incomplete* if some pairs of traders cannot trade directly with each other. Then trade between an initial seller i and a final buyer j involves a (possibly large) series of intermediate trades. The greater the incompleteness of the network, the more intermediation is required to achieve an efficient final allocation. Since intermediation takes time and time is costly, the costs of intermediation provide an important source of market friction. When the costs of intermediation are small, trade is executed quickly and efficiently. When the costs of intermediation are substantial, the cost and uncertainty of trade may give rise to other problems. In extreme cases, it can lead to a market breakdown.

The reasons for incompleteness are many. Asymmetric information may mean that traders will only deal with others that they know and trust. Even where information is symmetric, counter party risks may provide a role for an intermediary to act as a guarantor. Transaction costs and increasing returns may give one trader an incentive to limit the number of traders with whom he conducts business, or to force small traders to deal through an intermediary broker. All of these motives for incompleteness give rise to network architectures that may be efficient in normal times, but may also prove fragile when there is a sudden shock.

The theoretical work that is closest to ours is by Rachel E. Kranton and Deborah F. Minehart (2001), who study the interaction of a group of buyers and sellers who are asymmetrically connected with each other. Their market does not allow for intermediation, however. Intermediation is the central element of our model and does not appear to have been studied in the network literature.

Our objective is to begin the study of exchange in incomplete networks. We represent a network by a connected graph in which the nodes represent agents (traders) and the edges represent the possibility of trade between the agents linked by the edge. In the next section, we present a theoretical model of a simple asset market and sketch a proof that, in the limit as the period length goes to zero, the market becomes frictionless, and the market outcome is efficient. Then we discuss the application of the theory, first, to explain the occurrence of market breakdowns as a result of frictions and incomplete networks and, secondly, to investigate the consequences of market breakdowns on equilibrium prices and the allocation of assets. Finally, we discuss the possibility of using the model as the basis of an experimental

design that can be implemented in the laboratory.

I. A model of financial networks

In our proto-type model, agents bargain over terms of trade and exchange an indivisible asset according to a standard protocol. The model consists of a finite set of agents N and $k < N$ identical and indivisible units of an asset. (We abuse notation by letting N represent both the set and its cardinality). Time is divided into an infinite sequence of dates, indexed by t. A crucial feature of the network is that each node has a capacity constraint. For simplicity, we impose the capacity constraint by assuming that each agent $i \in N$ can hold at most one unit at any date. (An alternative is to assume that each agent has a limited budget for purchasing assets). The initial distribution of assets is denoted by $e = (e_1, ..., e_N) \in \{0,1\}^N$, where $e_i \in \{0,1\}$ is the number of units of the asset initially held by agent i. The financial network is represented by a non-empty graph (N, E), where N is the set of nodes and $E = \cup_{i=1}^N \{(i,j) : j \in N_i\} \subset N \times N$ is the set of edges. Thus, the set $N_i = \{j : (i,j) \in E\}$ denotes the neighbors of agent i, i.e., the set of agents with whom agent i can trade.

We assume that agents discount future utilities using a common discount factor $0 < \delta < 1$, so agent i receives a flow utility of $u_i = (1 - \delta)\bar{u}_i \geq 0$ from holding one unit of the asset, where $\bar{u}_i \geq 0$ is the present value of agent i's stream of utility from holding the asset forever. An agent who does not hold an asset during the period receives a flow utility of zero. Since each agent can hold at most one unit of the asset, an attainable allocation at any date is represented by a vector $x \in \{0,1\}^N$ such that $\sum_{i \in N} x_i = k$, where $x_i \in \{0,1\}$ is the number of units held by agent i. We let X denote the set of all attainable allocations. We assume that the non-zero flow utilities are generic, so without essential loss of generality we can order the agents such that $u_1 > u_2 > ... > u_{N'} > 0$ and $u_i = 0$ for $i = N' + 1, ..., N$. The first N' agents are called *investors* and the last $N - N'$ agents are called *intermediaries*. Note that we allow for the possibility that $N' = N$ but always assume that $N' > k$. In that case, there is a unique Pareto-efficient allocation $x^* = \arg\max \{u \cdot x : x \in X\}$. At the allocation x^* the sum of flow utilities — and hence the present value of the stream of flow utilities — is maximized in each period. Any departure from this state must produce a strictly lower present value of the stream of flow utilities, which would imply a violation of individual rationality for some agent. Hence, x^* is an absorbing state: if the process $\{x_t\}$ ever reaches the state x^*, it will remain there.

The trading protocol is defined by the following rules. For any attainable allocation $x \in X$, let $B(x)$ denote the set of buyers and let $S(x)$ denote the set of sellers when the initial allocation is x. At each date t, with initial allocation x_{t-1}, an agent $i \in S(x_{t-1})$ is chosen at random to be the proposer. Seller i makes a proposal to one of the buyers $j \in B(x_{t-1}) \cap N_i$, assuming the set is non-empty, by offering a price p at which he is willing to sell his unit of the asset to j. Buyer j accepts (A) or rejects (R) the proposal. If he accepts, the asset is transferred from agent i to agent j and j pays i the price p. Otherwise no trade occurs and the allocation remains the same at the beginning of the next period.

These rules define an extensive-form game of perfect information denoted by $\Gamma(\delta)$. We analyze this game using the *Markov perfect equilibrium* (MPE) as the solution concept. To describe the equilibrium path we need some additional notation. The Markov equilibrium strategy of agent i is denoted by f_i, where $f_i : X \rightarrow B(x) \times \mathbb{R}_+$ if $i \in S(x)$ and $f_i : X \times S(x) \times \mathbb{R}_+ \rightarrow \{A, R\}$ if $i \in B(x)$. Let $\phi_i(x)$ denote the allocation that results from x when $i \in S(x)$ is chosen as the proposer and let $\phi_i(x) = x$ if $i \notin S(x)$. Then the transition probability on the set of attainable allocations is defined by

$$P(x, x') = \sum_{i \in S(x)} \sum_{j \in B(x)} \pi_i(x) \chi_{j \times \mathbb{R}_+}(f_i(x)) \chi_A(f_j(x, f_i(x))) \chi_{x'}(\phi_i(x))$$

if $x \neq x'$ and $P(x, x) = 1 - \sum_{x' \neq x} P(x, x')$ otherwise. Let $\{f_i\}$ be a fixed but arbitrary MPE of the game $\Gamma(\delta)$ and let $v(x)$ be the equilibrium payoff. Then, the *Bellman equation* is given by

$$v_i(x) = u_i + \delta \pi_i(x) \max \left\{ v_i(x), \max_{j \in N_i \cap B(x)} v_j(\phi_{ij}(x)) \right\} + \delta \sum_{j \in S(x) \setminus \{i\}} \pi_j(x) v_i(\phi_j(x)),$$

if $x_i = 1$ and $v_i(x) = 0$ otherwise, where $\pi_i(x)$ denotes the probability that agent i is chosen as the proposer when the allocation is x and $\phi_{ij}(x)$ denotes the allocation obtained from x by transferring one unit from i to j.

The first question we address is "What happens in the limit as the time period becomes vanishingly small?" This corresponds to the case where there are no frictions and trades can occur infinitely fast. Formally, this is equivalent to letting the discount factor $\delta \rightarrow 1$ while holding constant the asset values $\bar{u}$. What follows is a sketch of the asymptotic analysis of equilibrium.

Proposition 1. *The Pareto-efficient allocation x^* is an absorbing state in any equilibrium.*

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To show this, let $\{\Phi_t\}$ denote the random path of equilibrium alloca-
tions, with $\Phi_0 \equiv e$. Then it follows from feasibility that, for any attainable
allocation x,

$$v(x) \cdot x \leq \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t u \cdot \Phi_t \,\middle|\, \Phi_0 = x\right] \leq \sum_{t=0}^{\infty} \delta^t u \cdot x^* = \bar{u} \cdot x^*$$

and the inequality is strict if $\Phi_t \neq x^*$ for some t with positive probability.
Now, starting at the allocation x^*, any $i \in S(x^*)$ can achieve an equilibrium
payoff $\sum_{t=0}^{\infty} \delta^t u_i = \bar{u}_i$ by holding the asset forever. Thus, $v_i(x^*) \geq \bar{u}_i$ and,
together with the preceding inequality, this implies that $v_i(x^*) = \bar{u}_i$ for all $i \in$
$S(x^*)$. It then follows from the first inequality that $\mathbf{P}\left[\Phi_t = x^*,\ \forall t\,\middle|\, \Phi_0 = x^*\right] =$
1 as required.

Proposition 2. *In any equilibrium, any absorbing set is a sin-
gleton, i.e., there are no limit cycles.*

To see this, call a set $A \subset X$ an *absorbing set* if it is a minimal set with
the property that $\mathbf{P}[\Phi_{t+1} \in A | \Phi_t \in A] = 1$. Note that once A is entered, each
element of A is reached infinitely often with probability one. Suppose that
the set A is not a singleton and note that as the process Φ cycles through the
elements of A, there must be more than k agents who hold units of the asset.
Let $S(A) = \cup_{x \in A} S(x)$ and index the elements of $S(A)$ by $i_1, i_2, ..., i_m$ so that
$u_{i_r} > u_{i_{r+1}}$ for $r = 1, ..., m - 1$ and note that by hypothesis $|S(A)| > k$.
Suppose that $x^1 \in A$ is an allocation such that i_1 holds a unit of the asset
so once x^1 is reached, i_1 will hold it forever. Then after x^1 is reached, there
must be an allocation x^2 in which i_2 holds a unit of the asset and once this
allocation is reached both i_1 and i_2 will hold their units forever. Eventually,
we must reach an allocation x^k where the agents $i_1, ..., i_k$ all hold the asset
and will never give it up. Then x^k is an absorbing state, contradicting the
definition of A.

Proposition 3. *In any equilibrium, the process $\Phi = \{\Phi_t\}$ must
reach an absorbing state with probability one.*

The proof is again by contradiction. Let $x^1, ..., x^m$ denote the absorbing
states and suppose that Φ does not reach an absorbing state with probability
one. Note that if Φ reaches an absorbing state with positive probability then
(because of the Markov assumption) it reaches an absorbing state with prob-
ability one. This implies that Φ reaches $\{x^1, ..., x^m\}$ with probability zero.

Then, our previous result implies that $X \setminus \{x^1, ..., x^m\}$ contains an absorbing state, contradicting the definition of $x^1, ..., x^m$.

Our main result is to show that, in the limit as $\delta \to 1$, the economy becomes frictionless and x^* is the only absorbing state for any MPE of $\Gamma(\delta)$. The answer turns out to depend on the key assumption that the efficient asset holders are "accessible." The efficient asset holders are said to be *accessible* if the intermediaries form a non-empty, connected network and every efficient asset holder is directly connected to at least one intermediary, i.e., for any $i = 1, ..., k$, there is some $j > N'$ such that $i \in N_j$.

> **Theorem:** *For δ sufficiently close 1, the only absorbing state is x^* if the efficient asset holders are accessible.*

To see this, suppose there exists an absorbing state $x \neq x^*$ and let $\bar{v}$ denote the lowest payoff of any asset holder in any absorbing state. Then any agent to whom this agent could pass the asset must have a reservation price less than or equal to $\bar{v}$ and by induction we can show the same must be true for any allocation and any agent to whom the asset could be passed directly or indirectly. It is crucial here, however, that there be no absorbing state in which any asset is ever worth less than $\bar{v}$ in the limit. To rule out this possibility, it is sufficient that the network satisfy the assumption of accessibility. Since we assume that $N' > k$, the intermediaries can never hold the asset in an absorbing state, and the existence of intermediaries without assets ensures that there is a path from any asset holder to any other asset holder. Then we can show the existence of an agent without an asset whose reservation price is greater than $\bar{v}$. This contradiction proves the desired result.

This heuristic argument skirts all the difficulties inherent in the passage to the limit as $\delta \to 1$. In fact, in several steps we can also show that when the efficient asset holders are accessible, there exists a number $0 < \delta_0 < 1$ such that the Pareto-efficient allocation x^* is the only absorbing state of any equilibrium of the game $\Gamma(\delta)$ with $\delta_0 < \delta < 1$.

II. Market breakdowns

One of the constant themes in the literature on financial markets is that financial markets, unlike many other markets, are *fragile*. Small shocks have large consequences in these markets. Examples of this phenomena are provided by Utpal Bhattacharya and Matthew I. Spiegel (1991); Allen and Gale (2000);

Freixas, Parigi and Rochet (2000); Yaron Leitner (2005); and Frederic Bois-say (2006). In each case, the source of fragility is different. In our model, the fragility of the markets arises from the interaction of several factors, limited carrying capacity, trading uncertainty, and costly discounting.

Our Theorem characterizes the MPE of markets that are nearly friction-less in the sense that trade occurs sufficiently quickly that we can ignore discounting. This asymptotic analysis provides some understanding of the role for intermediaries in helping markets to achieve efficiency. At least, it provides a benchmark by which we can judge markets that do not satisfy the assumptions needed for the theorem. The assumptions are very strong, however, and that really provides a motive for study markets where frictions are important.

There are several important frictions in the market: trading uncertainty, represented here by the random arrival of opportunities to make an offer; the opportunity cost of funds tied up in inventories of assets, represented here by the discount factor δ; and the costs of managing an inventory of assets, represented here by the unit capacity constraint. In normal times, limited capacities and low costs may be consistent with steady flows that clear the market, but in abnormal times, a small shock can destabilize the system. A reduction in capacity or an increase in discounting or trading uncertainty may cause individual traders to withdraw from active trading, resulting in further falls in market capacity and further increases in trading uncertainty. The result may be a market breakdown, in which the ability to trade in the market at any acceptable prize disappears.

A concrete example of how frictions, such as trading costs or discounting, can lead to *market breakdowns* will make the process clear. Suppose that there are five agents $i = 1, ..., 5$ arranged in line: i can trade with $i + 1$ for $i = 1, ..., 4$ and these are the only trades allowed. Initially, agents 1 and 3 are endowed with one unit of the asset and these are the only agents with an endowment of the asset. Only agents 3 and 5 value the asset: specifically, we assume that $\bar{u}_3 = 5 - \varepsilon$, $\bar{u}_5 = 5$, and $\bar{u}_i = 0$ for $i = 1, 2, 4$. The unique efficient allocation is $x^* = (0, 0, 1, 0, 1)$ and this will be the limit allocation if the discount factor δ is sufficiently close to 1. Agent 1 will sell his unit of the asset for approximately $5 - \varepsilon$ and agent 3 will sell his unit of the asset for approximately 5. Now consider what happens as the length of a trading period increases. This corresponds to reducing the value of δ, holding $\bar{u}$ constant. Trading the asset is becoming more costly. The most that agent 3 can hope to get for the asset is $\delta \times 5$ and it may be less because of the random

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selection of the proposer. If he sells the asset, he can purchase another unit of the asset but the price he will have to pay is determined by bargaining. In the case where all proposals are made by sellers, agent 2 will demand $5 - \varepsilon$ for the asset in any subgame perfect equilibrium. So, unless the price that agent 3 receives for his unit of the asset is greater than $5 - \varepsilon$, he must make a loss from trade. Then, for some value of $\delta > 0$, he will be indifferent between trading the asset and holding it in perpetuity and a further decrease in δ will cause a market breakdown, i.e., a discontinuous change in the equilibrium asset prices and allocations. The limiting allocation will be $x^* = (1, 0, 1, 0, 0)$ and agent 1 will not be able to sell his unit of the asset for any price greater than zero.

Other shocks that may lead to a market breakdown include a change in network architecture; a change in transaction costs; and a change in the value of the asset.

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