1. Facts on Circular Functions: Consider the trigonometric functions \( \cos(\omega) \) and \( \sin(\omega) \), where \( \omega \) is a real number giving the angle in radians. These functions are periodic, with \( \cos(\omega+2\pi k) = \cos(\omega) \), \( \cos(\pi k) = (-1)^k \), \( \sin(\omega+2\pi) = \sin(\omega) \), \( \sin(\pi k) = 0 \), and \( \cos(\omega) = \sin(\omega+\pi/2) \) for \( k = \pm1,\pm2,... \). Define the complex valued function \( \exp(\omega) = \cos(\omega) + i\sin(\omega) \), where \( i = (-1)^{1/2} \). Then \( \exp(\omega+2\pi) = \exp(\omega) \) and \( \exp(\pi k) = (-1)^k \). Here are some other useful relationships —

(1) \[ \cos(\omega) = \frac{e^{i\omega} + e^{-i\omega}}{2} \quad \text{and} \quad \sin(\omega) = \frac{e^{i\omega} - e^{-i\omega}}{2i} \]

(2) \[ \int_{-\pi}^{\pi} \cos(\omega k)d\omega = \int_{-\pi}^{\pi} \sin(\omega k)d\omega = \int_{-\pi}^{\pi} \exp(\omega k)d\omega = 0 \quad \text{for} \quad k = \pm1,\pm2,... \]

(3) \[ \int_{-\pi}^{\pi} \cos(0)d\omega = \int_{-\pi}^{\pi} \exp(0)d\omega = 2\pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(0)d\omega = 0 \]

(4) \[ \int_{-\pi}^{\pi} \cos(\omega k)^2d\omega = \int_{-\pi}^{\pi} \sin(\omega k)^2d\omega = \pi \quad \text{for} \quad k = \pm1,\pm2,... \]

(5) \[ \int_{-\pi}^{\pi} \exp(\omega k)\exp(-\omega k)d\omega = 2\pi \quad \text{for} \quad k = \pm1,\pm2,... \]

(6) \[ \int_{-\pi}^{\pi} \exp(\omega k)\exp(-\omega m)d\omega = 0 \quad \text{for} \quad k,m = 0,\pm1,\pm2,... \quad \text{and} \quad k \neq m \]

(7) \[ \int_{-\pi}^{\pi} \cos(\omega k)\cos(\omega m)d\omega = \int_{-\pi}^{\pi} \sin(\omega k)\sin(\omega m)d\omega = 0 \quad \text{for} \quad k,m = 0,\pm1,\pm2,... \quad \text{and} \quad k \neq m \]

(8) \[ \int_{-\pi}^{\pi} \cos(\omega k)\sin(\omega m)d\omega = 0 \quad \text{for} \quad k,m = 0,\pm1,\pm2,... \]
These formulas are found in handbooks of mathematical functions, and are demonstrated in textbooks on orthogonal polynomials or on Fourier analysis.

Suppose \( T > 1 \) is an integer, and define \( n = \lfloor T/2 \rfloor \), the largest integer satisfying \( n \leq T/2 \). Define the system of functions \( \psi_k(t) = (T)^{-1/2} \exp(i2\pi tk/T) \) for \( t = 1, \ldots, T \) and \( k = -n, -n+1, \ldots, 0, \ldots, n-1 \) for \( T \) even or \( k = -n+1, \ldots, 0, \ldots, n-1 \) for \( T \) odd.

Every complex-valued function \( h(t) \) can be written as \( h(t) = h_1(t) + i h_2(t) \) with \( h_1 \) and \( h_2 \) real-valued. The complex conjugate of \( h \) is \( h^*(t) = h_1(t) - i h_2(t) \), and the product \( h(t)h^*(t) = h_1(t)^2 + h_2(t)^2 \). Apply the formula for geometric sums to show that

\[
\sum_{k=-n}^{n-1} \psi_k(t) \psi_m^*(t) = \delta(k-m).
\]

Then the system of circular functions \( \psi_k(t) \) form an orthonormal basis for \( \mathbb{R}^T \). Suppose \( y_1, \ldots, y_T \) is a sequence of numbers, which may be deterministic or may be a realization from some stochastic process. This sequence can be represented in terms of the system of circular functions. Hereafter, assume \( T \) even and \( n = T/2 \). (Analogous formulas hold when \( T \) is odd, \( n = (T+1)/2 \), and the \( k = -n \) term in the sums below are dropped.) The relationship is

\[
y_i = \sum_{k=-n}^{n-1} \psi_k(t) x_k
\]

with

\[
x_k = \sum_{t=1}^{T} \psi_k^*(t) y_t.
\]

Verify that these formulas follow from the projection of \( (y_1, \ldots, y_T) \) on the space spanned by the vectors \( (\psi_k(1), \ldots, \psi_k(T)) \) for \( k = -n, \ldots, n-1 \); i.e., the regression of \( (y_1, \ldots, y_T) \) on these vectors. The vector \( (x_1, \ldots, x_T) \) is termed the Fourier representation of \( (y_1, \ldots, y_T) \). Write out the real and imaginary parts of (10) and (11) to get the equivalent formulas

\[
y_i = \sum_{k=-n}^{n-1} \cos(2\pi kt/T) a_k + \sum_{k=-n}^{n-1} \sin(2\pi kt/T) b_k
\]

with

\[
a_k = T^{-1} \sum_{t=1}^{T} \cos(2\pi kt/T) y_t \quad \text{and} \quad b_k = T^{-1} \sum_{t=1}^{T} \sin(2\pi kt/T) y_t.
\]

Show that \( \sum_{t=1}^{T} y_t^2 = \sum_{k=-n}^{n-1} x_k x_k^* \).
2. Suppose \( h \) is a real-valued function on an interval \([ -\pi, \pi ]\). For \( T \) a large even integer and \( n = T/2 \), define \( y_t = h(-\pi + 2\pi t/T)T^{-1/2} \). Let \( x_k \) be the Fourier coefficient given by (11), and define \( z_k = 2\pi e^{ikx_k} \). The Fourier representation of the sequence \( y_t \), from (11), is

\[
(14) \quad x_k = \sum_{t=-T}^{T} \psi^\ast(t)y_t = T^{-1} \sum_{t=-T}^{T} e^{i2\pi kn/T}h(-\pi + 2\pi t/T),
\]

implying

\[
(15) \quad z_k = \frac{2\pi}{T} \sum_{t=-T}^{T} e^{i2\pi kn/T}h(-\pi + 2\pi t/T)
\]

and, from (10),

\[
(16) \quad h(-\pi + 2\pi t/T) = \sum_{k=-n}^{n-1} e^{i2\pi kn/T}z_k/2\pi.
\]

Now let \( T \to \infty \). Suppose \( h \) is of bounded variation (i.e., can be written as the difference of two increasing bounded functions). Then it is continuous except at most at a countable number of points, and is square integrable. Then (15) converges to

\[
(17) \quad z_k = 2\pi \int_{0}^{1} e^{i2\pi ks+\pi kn}h(-\pi + 2\pi s)ds.
\]

A further change of variable to \( r = -\pi + 2\pi s \), implying \( -12\pi k s + \pi kn = -1kr \), yields

\[
(18) \quad z_k = \int_{-\pi}^{\pi} e^{i\pi r}h(r)dr.
\]

Show that the \( z_k \) satisfy

\[
\sum_{k=-n}^{n-1} z_k^\ast(z_k) = (4\pi^2/T) \int_{-\pi}^{\pi} h(-\pi + 2\pi t/T)^2 \to 2\pi \int_{-\pi}^{\pi} h(r)^2dr.
\]

Then, the limit of (16), evaluated at \( t = [T(r+\pi)/2\pi] \), as \( n \to \infty \) exists for \( r > -\pi \) and equals

\[
(19) \quad h(r) = \sum_{k=-\infty}^{+\infty} e^{ikr}z_k/2\pi
\]

at all continuity points of \( h \). The pair (18) and (19) give a Fourier representation of a function on
a bounded interval. If the function is periodic with \( h(r \pm 2\pi) = h(r) \) for all \( r \), then the Fourier representation holds for all \( r \). Using orthogonality properties of \( e^{ikr} \), show directly that if \( z_k \) is a square summable sequence, then applying (19) then (18) reproduces the sequence. Note that if \( h(z) \) is a sum of sines and cosines with frequencies that are multiples of \( 1/2\pi \), then the Fourier representation will have non-zero \( z_k \)'s only for the \( k \)'s corresponding to these frequencies. Then, the \( z_k \) series may be thought of as extracting the frequencies appearing in \( h(r) \).

3. Suppose \( h(r) \) is a square integrable real-valued function on the real line. For a large constant \( M \), apply the Fourier representation in the previous question to the function \( M \cdot h(Mr) \) for \(-\pi \leq r \leq \pi \) to obtain (18) and (19). Define a variable \( \omega = k/M \), or \( k = \omega M \), and a function \( H_M(\omega) \) on the real line by

\[
(20) \quad H_M(\omega) = \int_{-\pi M}^{+\pi M} e^{-i\omega s} h(s) ds \quad \text{or} \quad \int_{-\infty}^{+\infty} e^{-i\omega s} h(s) ds.
\]

Note that \( z_k = H(\omega) = \int_{-\pi M}^{+\pi M} e^{i\omega s} h(s) ds \), so that (19) can be written

\[
(21) \quad h(Mr) = \frac{1}{2\pi M} \cdot \sum_{k=-\infty}^{+\infty} e^{ikr} H_M(k/M).
\]

Letting \( s = Mr \) and \( \omega = k/M \), the limit of (21) as \( M \to \infty \), if it exists, becomes

\[
(22) \quad h(s) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{i\omega s} H(\omega) d\omega.
\]

The pair consisting of (22) and

\[
(23) \quad H(\omega) = \int_{-\infty}^{+\infty} e^{i\omega s} h(s) ds
\]

are *Fourier transforms*. This construction shows that Fourier transforms are obtained as limits of Fourier representations, and also shows that when the limits exist, the Fourier representations from Question 1 can be used to approximate the Fourier transforms. Show that if (22) and (23) are satisfied, then
\[ (24) \quad \int_{-\infty}^{+\infty} h(s)^2 ds = \int_{-\infty}^{+\infty} H(\omega)H^*(\omega) d\omega. \]

4. For the Fourier transforms (22) and (23), verify the following conditions:
   (1) \( h \) even implies \( H \) real and even
   (2) \( h \) odd implies \( H \) imaginary and odd
   (3) [time scaling] for \( c > 0 \), \( h(cs) \) transforms to \( c^{-1}H(\omega/c) \)
   (4) [frequency scaling] for \( c > 0 \), \( H(c\omega) \) transforms to \( c^{-1}H(s/c) \)
   (5) [time shifting] \( h(s-\tau) \) transforms to \( H(\omega)e^{-j\omega\tau} \)
   (6) [convolution] if \( g \) and \( h \) are real functions and \( G \) and \( H \) are their transforms, and if
      \[ (g*h)(s) = \int_{-\infty}^{+\infty} g(t)h(s-t)dt, \]
      then the transform of \( g*h \) is \( G(\omega)H(\omega) \).
   (7) [covariation] if \( g \) and \( h \) are real functions and \( \text{cov}(g,h) = \int_{-\infty}^{+\infty} g(s)h(s)ds \), then \( \text{cov}(g,h) \)
      \[ = \int_{-\infty}^{+\infty} G(\omega)H^*(\omega)d\omega. \]
   (8) [Parseval's theorem] \[ \int_{-\infty}^{+\infty} h(s)^2 ds = \int_{-\infty}^{+\infty} H(\omega)H^*(\omega) d\omega. \]