Existence of Walrasian Equilibrium

**Theorem** (Grandmont-McFadden, 1972)

Define the closed unit simplex \( U^* = \{ p \in \mathbb{R}^m \mid p \geq 0 \text{ and } 1 \cdot p = 1 \} \) and the open unit simplex \( U^0 = \{ p \in U^* \mid p \gg 0 \} \). Suppose there exists a set \( U \) with \( U^0 \subset U \subset U^* \) and an excess demand correspondence \( \zeta \) that maps \( U \) into non-empty subsets of \( \mathbb{R}^m \) and satisfies

(a) \( \zeta \) is bounded below; i.e., there exists \( b \in \mathbb{R}^m \) such that \( b \leq x \) for all \( x \in \zeta(p), p \in U \);

(b) For each \( p \in U \), \( \zeta(p) \) is a convex set, and \( p \cdot x \leq 0 \) for all \( x \in \zeta(p) \);

(c) \( \zeta \) is upper hemicontinuous on \( U \); i.e., the graph \( \{(p,x) \in U \times \mathbb{R}^m \mid x \in \zeta(p)\} \) is a closed subset of \( \mathbb{R}^m \times \mathbb{R}^m \).

Then there exists a \( p^* \in U \) and a \( x^* \in \zeta(p^*) \) such that \( x^* \leq 0 \).

**Proof:** Let \( U^k = \{ p \in U^0 \mid p \geq (1/mk,\ldots,1/mk) \} \); then the \( U^k \) are convex and compact and their union is \( U^0 \). Let \( X^k \) denote the closed convex hull of \( \{ \zeta(p) \mid p \in U^k \} \). Property (a), property (b) that \( p \cdot x \leq 0 \) for all \( x \in \zeta(p) \), and the definition of \( U^k \) imply that \( X^k \) is bounded, and hence compact. For \( (x,p) \in X^k \times U^k \), define a mapping \( \eta \) into non-empty subsets of \( X^k \times U^k \) by

\[
\eta(x,p) = \{(x',p') \in X^k \times U^k \mid x' \in \zeta(p) \text{ and } p' \cdot x \geq p'' \cdot x \text{ for all } p'' \in U^k\}.
\]

The maximands of a linear function \( p'' \cdot x \) on the compact convex set \( U^k \) form an upper hemicontinuous, convex valued correspondence. Together with properties (b) and (c) of \( \zeta \), this implies that \( \eta \) is an upper hemicontinuous, convex valued correspondence on \( X^k \times U^k \).

A fixed point theorem of Kakutani (1941) then guarantees that there exists \( (x^k,p^k) \) such that \( (x^k,p^k) \in \eta(x^k,p^k) \). Then \( x^k \in \zeta(p^k) \) and \( 0 \geq p^k \cdot x^k \geq p \cdot x^k \) for all \( p \in U^k \). Consider the sequence \( (x^k,p^k), k = 1,2,\ldots \). Property (a) and \( p^1 \cdot x^k \leq 0 \) imply that this sequence is bounded. Hence, it has a subsequence converging to a limit point \( (x^0,p^0) \). Property (c) implies \( x^0 \in \zeta(p^0) \), while the property \( 0 \geq p^k \cdot x^k \) for all \( p \in U^k \) implies \( 0 \geq p^0 \cdot x^0 \) for \( p \in U^0 \), since each \( p \in U^0 \) is contained in \( U^k \) for \( k \) sufficiently large. This in turn implies \( 0 \geq x^0 \). \( \square \)