Chapter II.4

FLEXIBILITY VERSUS EFFICIENCY IN *EX ANTE* PLANT DESIGN

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1. Introduction

Operating flexibility is an important attribute of fixed plant and equipment utilized in a production process, and a factor in the economics of capital–equipment design. The cost of adding flexibility is usually a loss in economic efficiency relative to a "best practice" design for a specific static operating environment. Consequently, the flexibility–efficiency margin is an intrinsic part of the economic calculus of the firm, and a factor to be weighed in econometric analysis of a firm's technological possibilities and behavior. In this paper we develop a model of the firm that incorporates in the plant-design decision a recognition of the possibilities for a tradeoff between flexibility and efficiency. We also provide an algorithm for generating econometric net supply systems within which this phenomenon can be studied empirically.

The following examples illustrate the role of the flexibility–efficiency decision in production processes, and indicate possible applications of the model developed in this paper.

Example 1. Electric utilities are required to meet a system demand that varies over time in a known cycle, and they do so by constructing a mix

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of base-load and peak-load generating units. Base-load units have higher capital cost, but lower operating cost, and represent the lower cost technology for providing continuous output. The optimal mix of generating units balances the economic efficiency of supplying the average demand against the flexibility of response to demand variation.

Example 2. In areas where variation in oil, gas, and coal prices causes the least-cost fuel type to vary over time, electric utilities often install boilers in thermal power plants which can be converted to use any of these fuels. These boilers increase capital and maintenance costs and thus result in inefficient production if only one type of fuel is used throughout the lifetime of the plant.

Example 3. In the construction of commercial and industrial buildings, heating and electrical systems are often installed with conversion features to facilitate future expansion or remodeling. The increase in initial cost is justified by uncertainty about eventual use of the structure.

Example 4. Most manufacturing and distribution processes require inventories of various goods due to uncertainties in demand. Firms increase flexibility in meeting demand variations by increasing inventories. However, the increase in inventory carrying cost lowers the economic efficiency of meeting average demand. Similarly, choice of a flexible design as a response to demand uncertainty can be seen in the number of product lines (models, brands) carried in retail stores and the number of commodities produced by multiple-output manufacturers.

Example 5. In multiple-stage production processes, the early stages may be designed so that their output can be tailored to alternative specifications in the latter stages. For example, an automobile chassis is designed so that it can be used in several model types – station wagons, sedans, etc. Consequently, it may not be the least-cost chassis for a single model type – say, station wagons. Flexibility in the production of model types is achieved at the expense of a loss of efficiency in the production of a single model type. Agriculture provides several other examples of this phenomenon: corn may be planted in a pattern that provides maximum yield when chopped for silage; in a second pattern that provides maximum grain yield; or in a third flexible pattern so that it can be either chopped or grown to maturity, depending on the corn price at harvest, with some sacrifice in yield. A similar effect occurs in the selection of breeds of pigs and chickens.

2. Historical Background

Economic theory has traditionally recognized that in the *ex ante* design of a plant, a firm can be expected to weight static efficiency versus dynamic flexibility, the latter being emphasized when the plant is expected to face a variable or uncertain environment. The following classic geometric argument was given by Stigler (1939). Remarkably, this argument has not been quantified and adapted to empirical analysis in the general case.

Consider the textbook *ex ante–ex post* cost curves illustrated in Figure 1(a). The firm can choose *ex ante* a plant design yielding one of the *ex post* average cost curves $EPAC_1$, $EPAC_2$, etc. The curve $EAAC$ is the *ex ante* envelope of such *ex post* curves, and each design is efficient (i.e., tangent to the *ex ante* envelope) at a single output. The curves $EPMC_1$, $EPMC_2$, etc. are *ex post* marginal cost curves, and $EAMC$ is the marginal curve for the *ex ante* envelope. At an output price $p = 4$, anticipated with certainty by the firm, the design $EPAC_1$ will be chosen. Figures 1(b) and 1(c) illustrate the same behavior in terms of total cost and profit. Turn now to the case in which variation in output

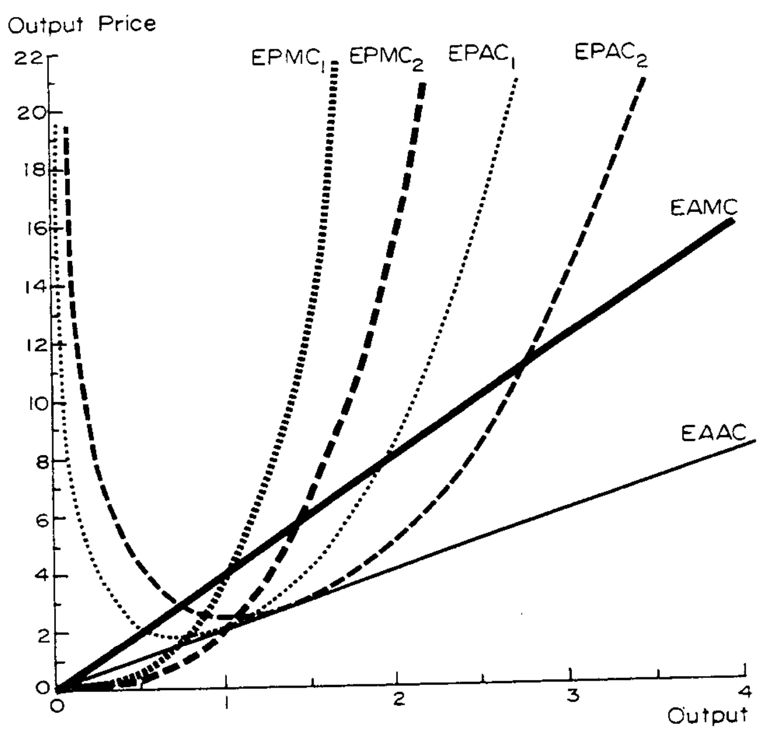

FIGURE 1(a)

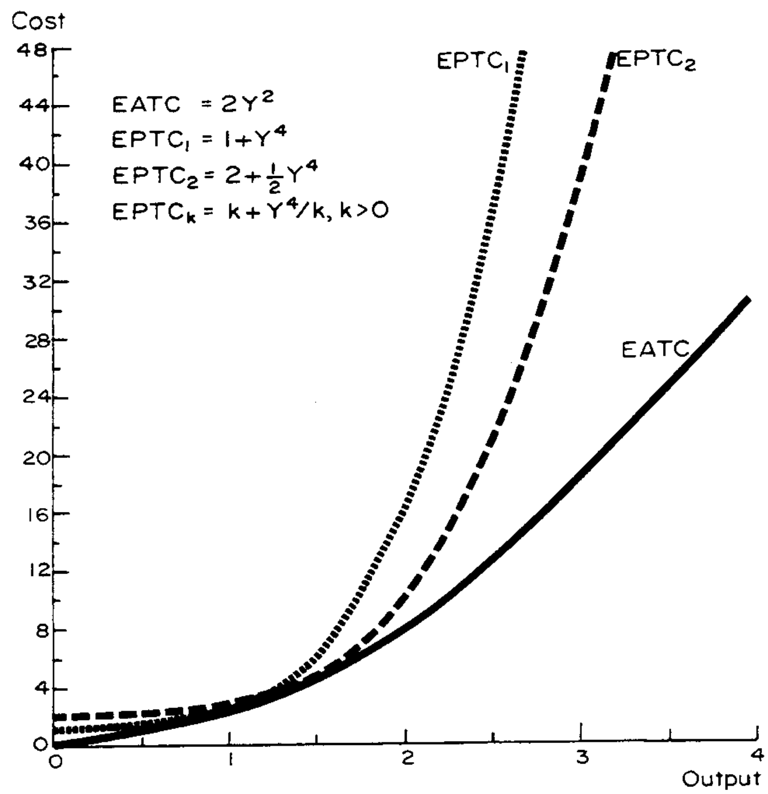

FIGURE 1(b)

price is anticipated, either because of uncertainty or because of intertemporal demand variation for the services of a durable plant. As illustrated in Figure 2(a), there may be a flexible plant design with *ex post* average cost curve $EPAC_3$ which is not efficient at any single output, but which may be a least-cost design given output price variability. For example, if output prices $p = 0$ and $p = 8$ are each anticipated with probability one-half, then this design can be seen in Figure 2(c) to yield higher expected profit than either of the statically efficient designs $EPAC_1$ or $EPAC_2$; i.e., the expected profit 3.0 from the third *ex post* technology, given by the abscissa of chord 3 at the expected price $p = 4$ exceeds the expected profit 2.77 from *ex post* technology one or 2.8 from *ex post* technology two (given by the mid-points of chords 1 and 2, respectively). It follows from the general property of convexity of profit as a function of output price that the expected profit from each of these technologies exceeds the corresponding profit associated with the expected output price.

One extreme case of the *ex ante–ex post* production structure illus-

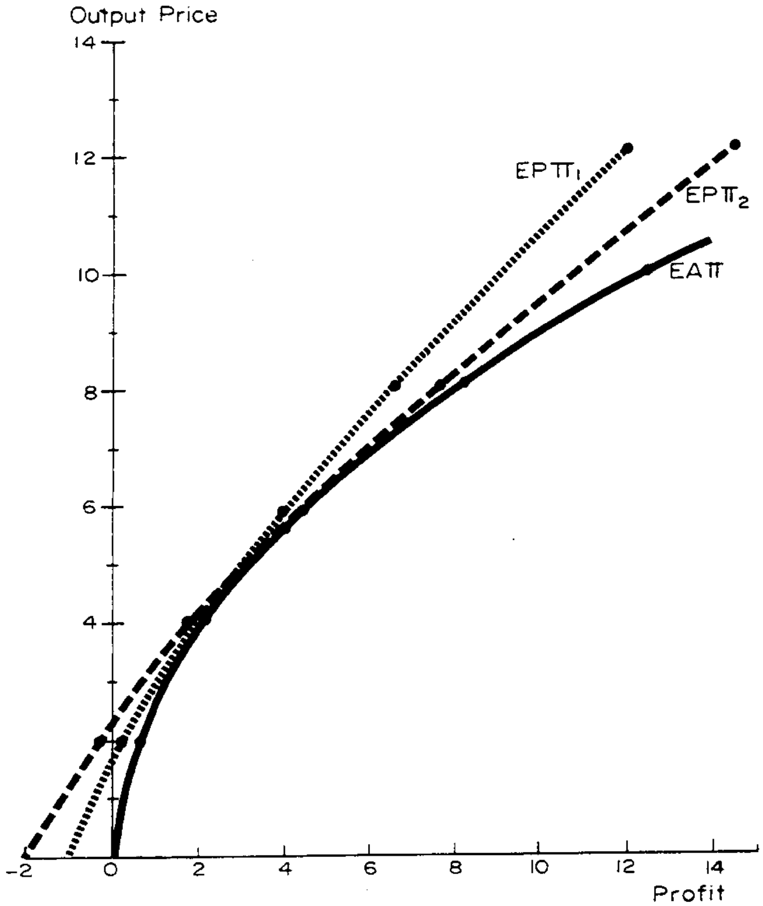

FIGURE 1(c)

trated in Figures 1 and 2 is the "putty-putty" model in which the envelope curve *EAAC* is itself a possible *ex post* design. Clearly, in this case *EAAC* will be the optimal design, achieving both maximum efficiency and flexibility. A second extreme case, illustrated in Figure 3, is the strict "putty-clay" model in which an *ex ante* design that achieves static efficiency at some output fixes the quantities of both capital and variable inputs in the *ex post* technology, and the only source of flexibility is free disposal of output. A less rigid "putty-clay" model in which variable inputs are required *ex post* in fixed proportion to capital services, but free disposal of capital services is possible, is illustrated in Figure 4. The class of intermediate cases between the "putty-putty" and "putty-clay" models, as illustrated in Figure 1, will be called "putty-semiputty" models. Note that the possibility of substituting flexibility for efficiency in the "putty-semiputty" model may be

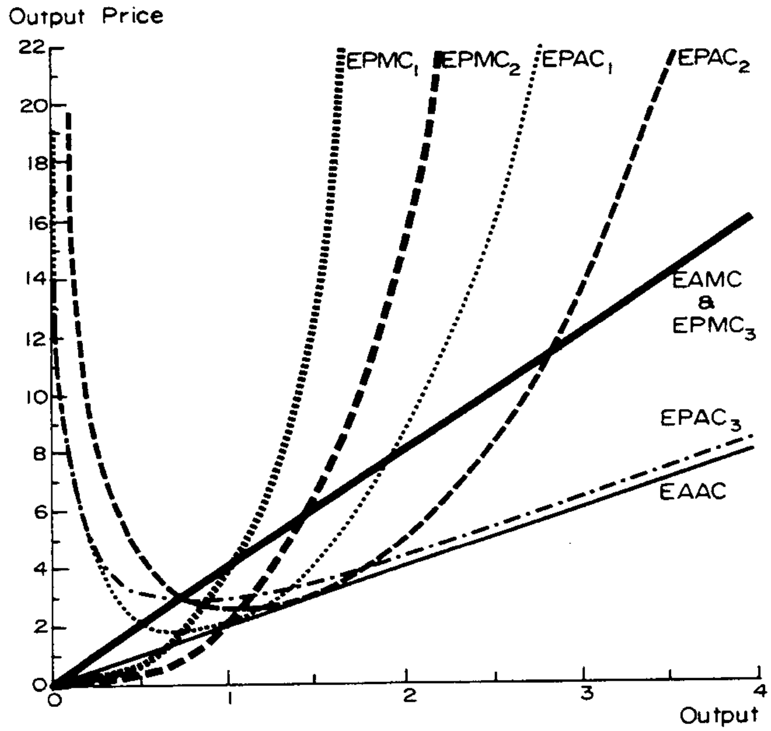

FIGURE 2(a)

present (Figure 2) or absent (Figure 1). Finally, note that the designs available to the firm may offer alternative types of flexibility as well as a simple flexibility–efficiency tradeoff. Figure 5 illustrates a case in which the *ex ante* options are an *ex post* technology that is flexible "upward" (output > 1) and one that is flexible "downward" (output < 1).[1]

The possibility of this flexibility–efficiency tradeoff in plant design has been largely ignored in econometric estimation of production functions,

[1]One footnote on the geometry of *ex ante–ex post* cost curves is in order. We first remind the reader of the geometric property established in the famous Wong–Viner footnote: The mutual tangency of an *ex ante* envelope and *ex post* average cost curve occurs at the output at which the *ex ante* and *ex post* marginal cost curves intersect (for example, unit output in Figure 1), and identifies the *ex ante* optimal *ex post* technology for the production of this static output. However, this output will not coincide with minimum *ex post* average cost unless the mutual tangency is horizontal. Next, note that when free disposal of output is possible, the total cost curve is monotone non-decreasing, implying that the elasticity of *ex post* average cost with respect to output is at least minus one; i.e., the negatively sloped leg of an average cost curve cannot rise more rapidly than a rectangular hyperbola. Thus, Figure 3 illustrates the most extreme possible case, and a diagram such as Figure 6 is impossible unless disposal of output is costly and *ex post* marginal cost is negative. This minor point is missed in many textbooks.

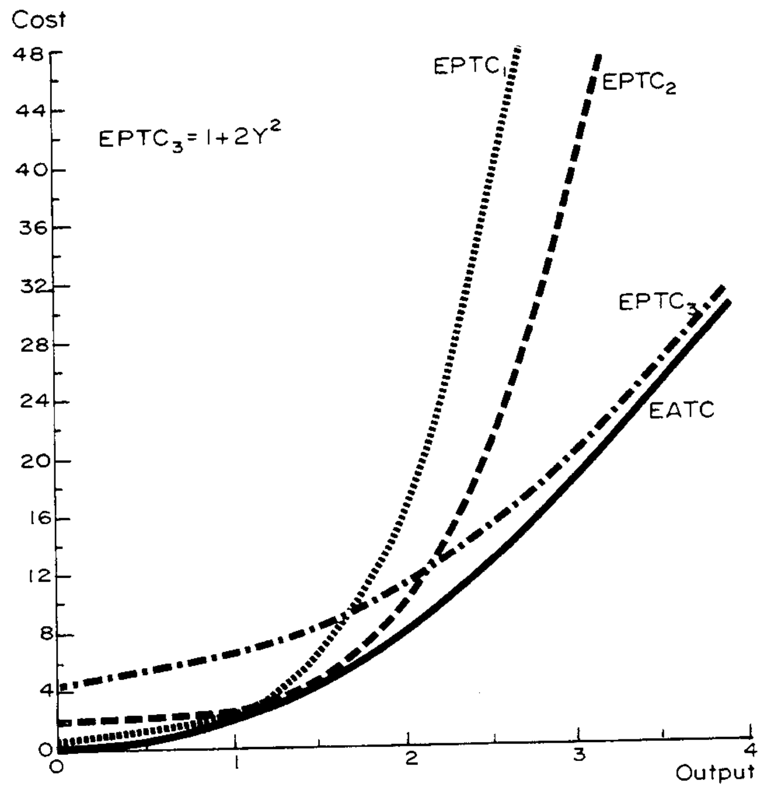

FIGURE 2(b)

probably due to the difficulty of quantifying the effect. In the "putty–clay" model, an attempt has been made by Attiyeh (1967) to obtain estimates when *ex post* factor price ratios may deviate from the factor price ratio for which the *ex post* design is efficient. However, all studies that have come to our attention assume that except for random errors, observed operating points lie on a locus of efficient production plans – an *ex ante* frontier for a cross-section study, or a particular *ex post* frontier for a time-series study. These frontiers are sometimes linked by an implicit "putty–putty" assumption, particularly in combined cross-section time-series studies. On the other hand, it should be clear that if the flexibility–efficiency tradeoff in plant design is present, the meaning of econometric production functions is altered substantially. This point can be developed most easily using Figure 7, which illustrates a family of *ex post* unit isoquants I_1, I_2, etc. and an envelope of efficient points E, conventionally interpreted as the *ex ante* unit isoquant. If firms anticipate variations in relative input prices, they may choose an *ex post* technology like I_3 or I_4. Observed operating points – say, v_1, v_2,... – will

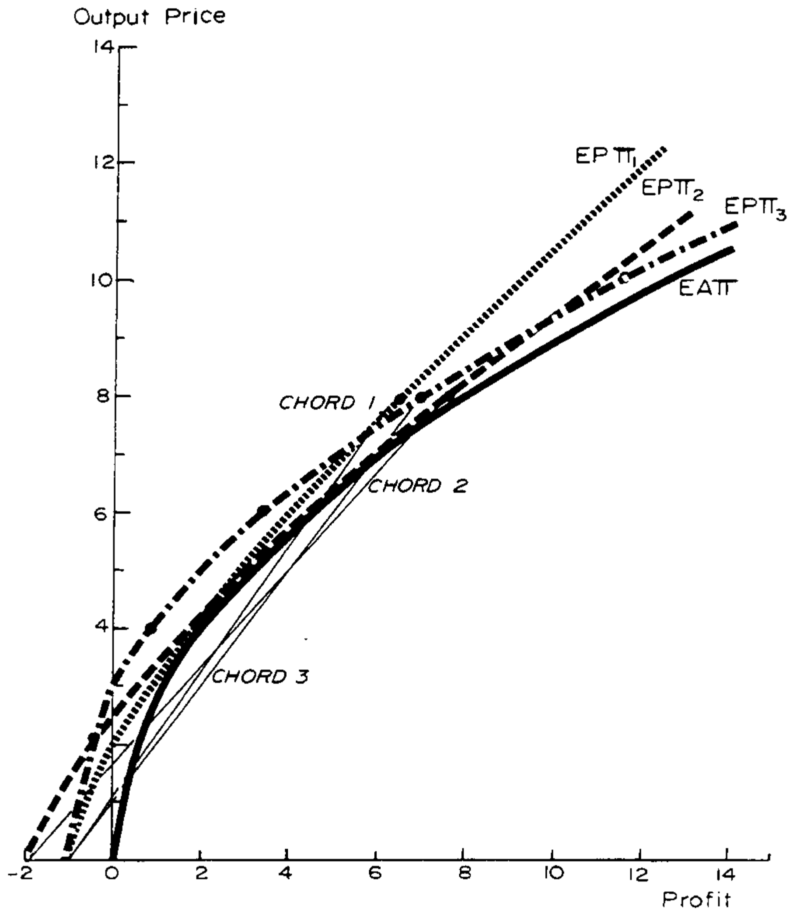

FIGURE 2(c)

lie on these curves. Suppose an econometric production function is fitted to these points. First, note that the unit isoquant of this function will typically underestimate substantially the efficiency of the "best practice" *ex ante* envelope E. This will be the case even if the estimated isoquant is taken to be the southwest boundary of the convex hull of the points v_1, v_2,... in order to reduce "optimization errors of the firm". Second, note that the points v_1, v_2,... may show considerable dispersion about the estimated isoquant, suggesting large measurement and optimization errors that are not in fact present. Third, note that the curvature of the estimated isoquant will bear no simple relation to the curvature of the envelope E. For example, in Figure 7 the utilization of data points like v_1, v_2, v_3, v_4 will produce an estimated isoquant with substantial curvature, whereas the envelope E has zero curvature.

The conclusion we draw from these observations is that, in the

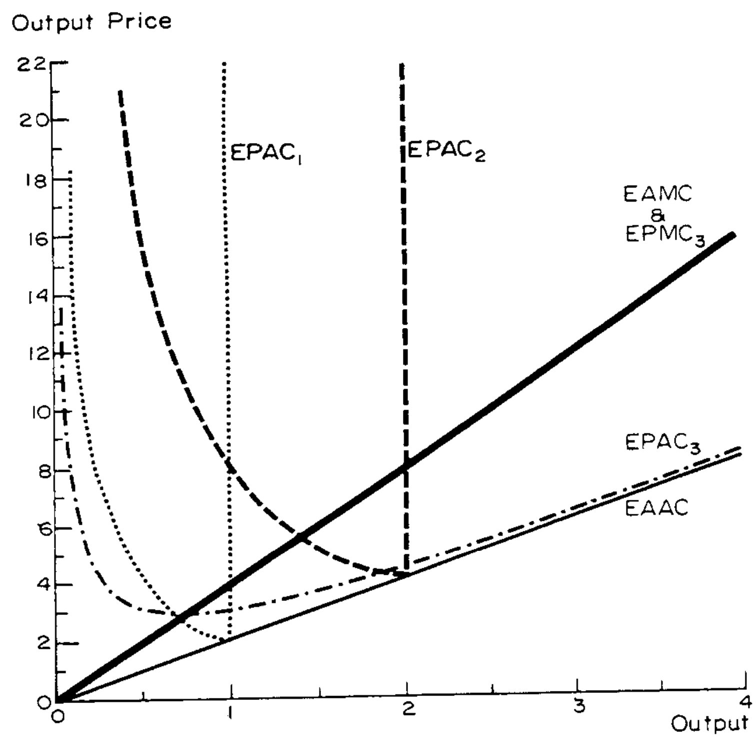

FIGURE 3(a)

presence of a significant flexibility–efficiency tradeoff, conventional econometric production functions provide very little information on the structure of the *ex ante* "best practice" envelope E, and may indeed provide misinformation. More fundamentally, we conclude that the concept of a static "best practice" envelope E characterizing the *ex ante* technology is inadequate, and in environments where firms face considerable uncertainty and intertemporal variation, irrelevant. It is in this case impossible to define meaningful isoquants, either theoretically or empirically, in a static picture of "one-period" production possibilities in which the flexibility–efficiency tradeoff has no explicit representation. The most satisfactory procedure would seem to be to abandon the elusive concept of the static *ex ante* isoquant and seek a quantification of *ex ante* design possibilities in which the total *ex post* operation of the plant is considered in the flexibility–efficiency decision.

Reliance on the concept of a static isoquant can also give rise to misleading conclusions in analyzing applied problems. For example, in the field of public utility regulation, consider the controversial Averch–Johnson (1962) thesis that a firm under a lax regulatory constraint may

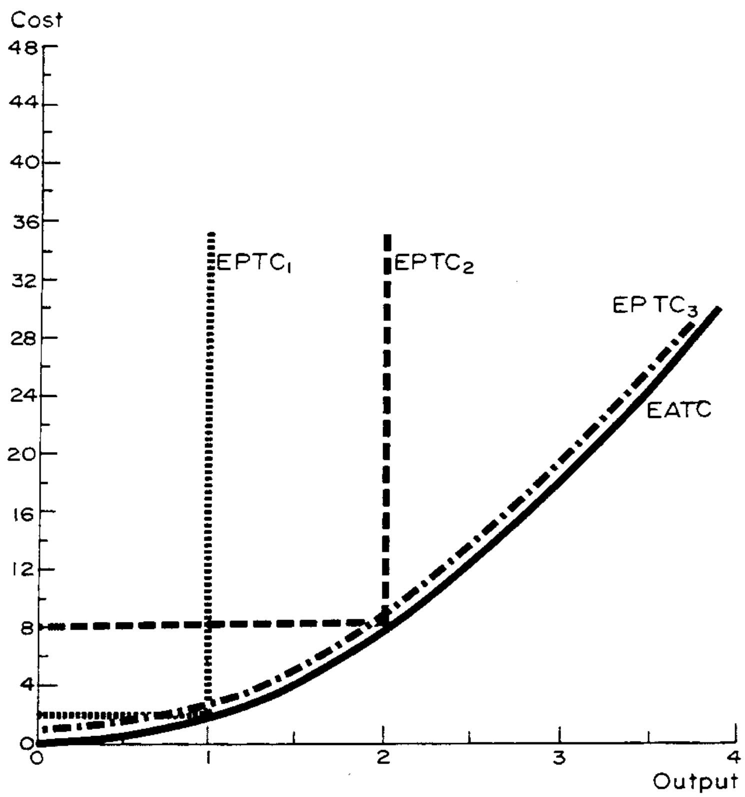

FIGURE 3(b)

overcapitalize and therefore operate inefficiently. Their analysis is based on the static isoquant. This isoquant is irrelevant for the majority of regulated industries, whose factors of production include a large percentage of durable capital with a long lifetime. Faced with intertemporal variation and uncertainty, cost-minimizing firms of this type will trade static efficiency for flexibility and, misleadingly, may appear inefficient under the Averch–Johnson analysis. The effect of regulatory constraint on investment and similar problems should be analyzed within the more general model presented in this paper when data are drawn from samples characterized by substantial shifts from "normal" values. An obvious example is post-1973 energy data.

3. A Model of the Firm with an *Ex Ante–Ex Post* Technology

We now present a model of the firm in which an *ex ante* decision is made on plant design, based on expectations about the environment to

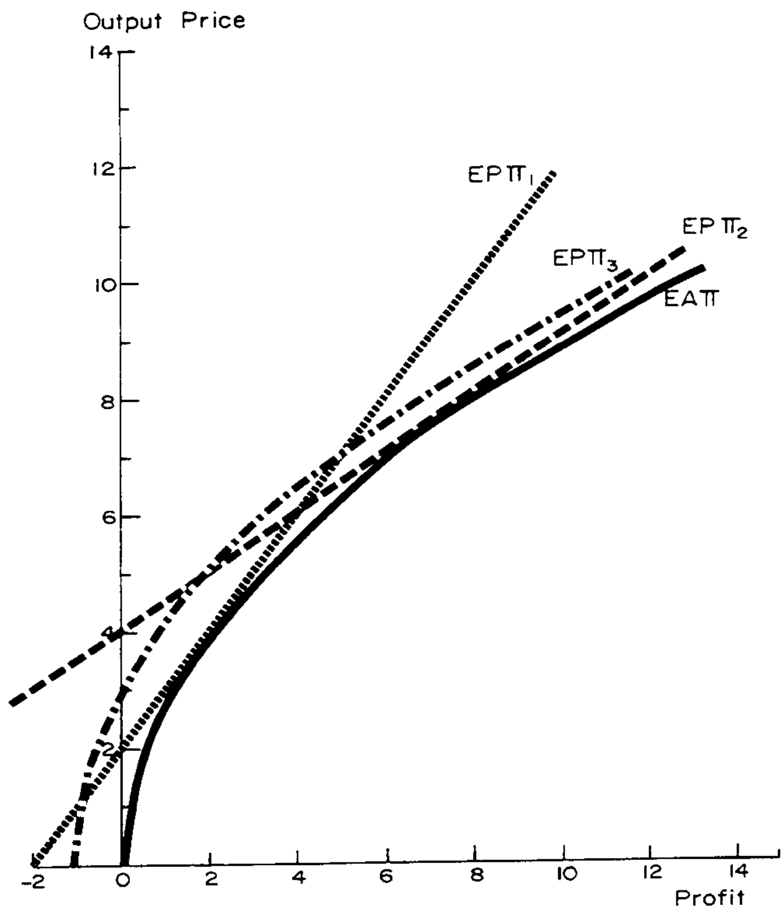

FIGURE 3(c)

be faced *ex post*; and then the resulting *ex post* technology is operated, taking into account the environment which actually prevails. We assume initially a simple choice structure in which a single *ex ante* design decision is made, followed by *ex post* operating decisions that are conditional on the design chosen. This structure will be appropriate for the case of a firm considering the construction of a durable plant that faces intertemporal variability, but no uncertainty. Alternately, it will be appropriate for the formally equivalent case of a firm that faces an *ex ante* design decision under uncertainty, and an *ex post* operating decision with no intertemporal variation.[2] It is possible to show that

[2]This formal equivalence is analogous to the equivalence that exists in consumer theory between the state preference analysis and the intertemporal allocation under certainty analysis when commodities in either different states of nature or different time periods are considered to be distinct.

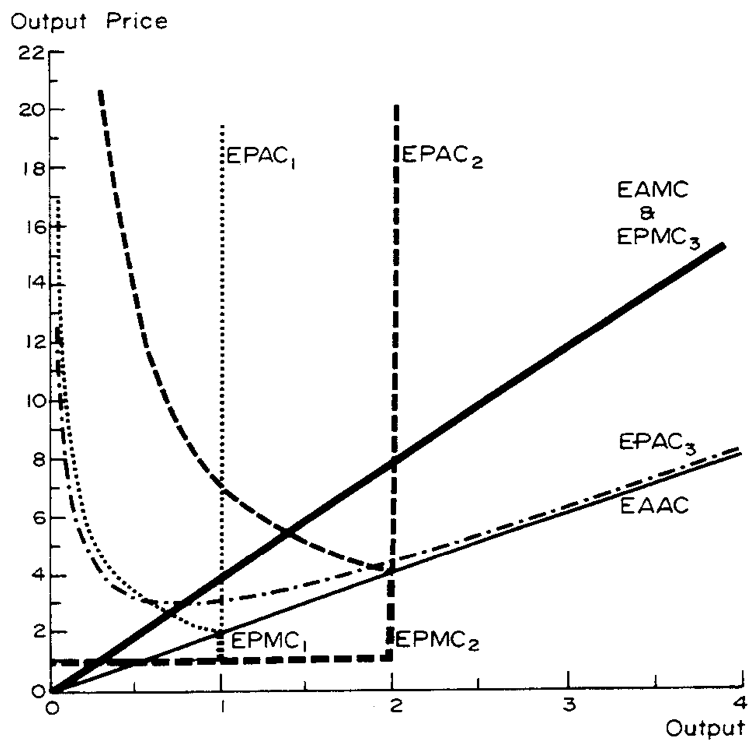

FIGURE 4(a)

more complex multiple-level choice structures (for example, involving construction decisions over time under uncertainty) can be formulated as dynamic programming problems in which the state equation corresponds to the two-stage choice structure described here.

We consider a firm that faces a set **S** of states of the future. A state s in **S** will have one of the following interpretations:

(a) The firm faces an intertemporal future without uncertainty, with $s \in \mathbf{S}$ denoting a chronological time and **S** denoting the set of times in which the plant under consideration might operate. **S** may be a set of discrete times or a continuum and may extend over a finite life-time or the entire future.

(b) The firm faces a one-period future with uncertainty, with $s \in \mathbf{S}$ denoting a state of nature, observed with the *ex ante* design decision but before the *ex post* operating decision. The set **S** of possible states of nature may be finite or infinite.

(c) When both intertemporal variation and uncertainty are present, $s \in \mathbf{S}$ may index both chronological time and state of nature. This case is

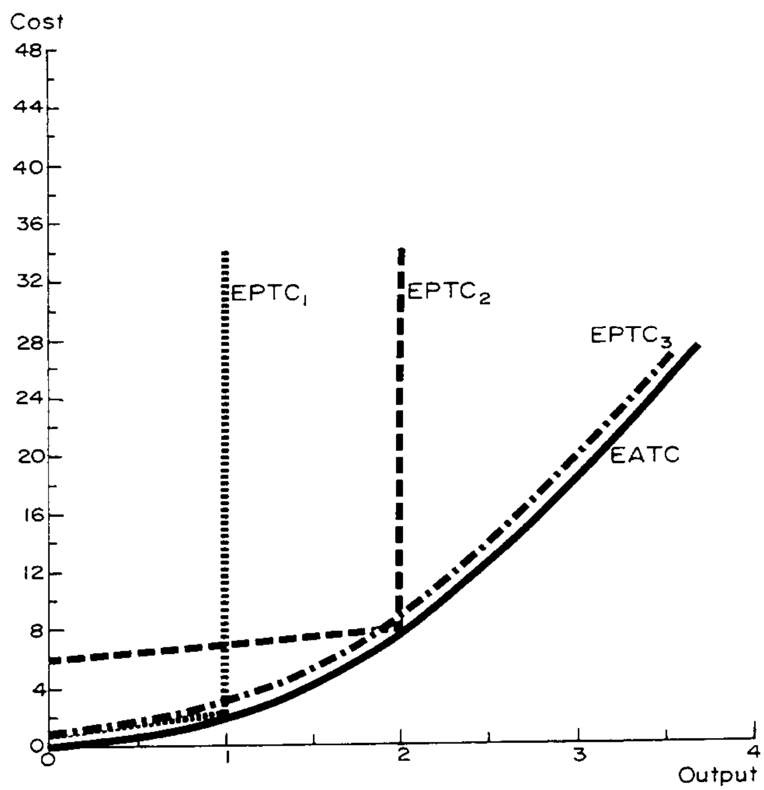

FIGURE 4(b)

formally equivalent to the previous ones, provided the assumption can be made that the firm receives no information in the course of operation that would induce a third-stage choice (and hence make necessary *ex ante* consideration of the strategic possibilities in this reconsideration).

The appropriateness of these three descriptions of the set of future states depends on the way producers form expectations. If a producer has a myopic time horizon but is uncertain about which state of nature will occur in the next period, description (b) is appropriate. If a producer has a planning horizon extending over several time periods but expects that one time sequence of states will occur with probability one and all others with probability zero, description (a) is appropriate. Description (c) applies when a producer forms expectations over a multiple-period time horizon and is uncertain about which state of nature will occur in each future time period, *provided* that the technological possibilities in each time period are independent of operating decisions in prior time periods (given the *ex ante* design) and that the producer does not revise

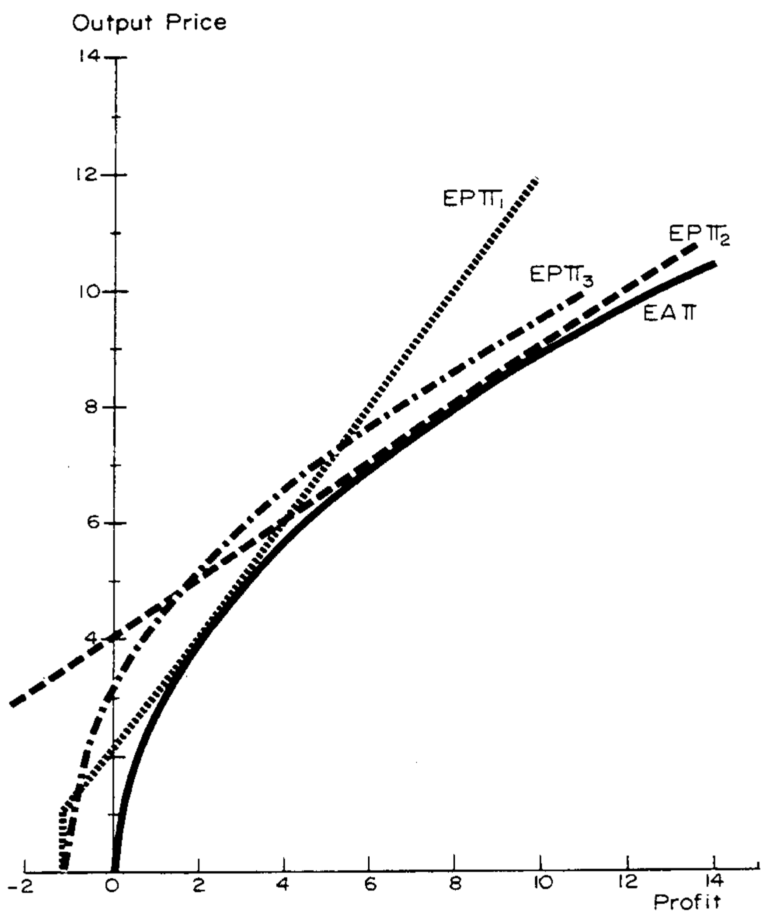

FIGURE 4(c)

his expectations in light of operating experience as time goes on. (This case is discussed in further detail in Section 6.)

Ex ante, the firm has available a set **B** of possible plant designs, with $\mathbf{b} = (\mathbf{a},\mathbf{K}) \in \mathbf{B}$ specifying an abstract vector **a** that describes plant layout, management organization, and exogenous variables influencing *ex post* operation, and a vector $\mathbf{K} = (K_1,...,K_J)$ of inputs of structures and fixed capital equipment.

Ex post, the firm faces competitive markets for N commodities, indexed $n = 1,...,N$, in each future state s, and the plant under consideration will supply a net output (netput) vector $\mathbf{x}_s = (x_{1s},...,x_{Ns})$ to these markets. A component x_{ns} is positive (negative) if commodity n is an output (input). The vector $\mathbf{x} = (\mathbf{x}_s : s \in \mathbf{S})$ is termed an *ex post production plan*. We emphasize that the commodities in the *ex post* production plan are those for which competitive spot markets exist, and hence are identified as "variable" netputs in the usual terminology.[3]

Choice of a plant design $\mathbf{b} = (\mathbf{a},\mathbf{K}) \in \mathbf{B}$ determines an *ex post variable technology* $\mathbf{V}(\mathbf{b})$, the set of *ex post* production plans that are possible for a plant with design **b**. Define a corresponding *ex post total technology* $T(\mathbf{a},\mathbf{K}) = \{(\mathbf{a},\mathbf{K},\mathbf{x}) | \mathbf{x} \in \mathbf{V}(\mathbf{a},\mathbf{K})\}$ for $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$. The set of all designs and *ex post* production plans available defines an *ex ante envelope technology* $\mathbf{T}^{ea} = \{(\mathbf{a},\mathbf{K},\mathbf{x}) | (\mathbf{a},\mathbf{K}) \in \mathbf{B},\ \mathbf{x} \in \mathbf{V}(\mathbf{a},\mathbf{K})\}$.

For the classical case of a firm that faces a non-varying future with certainty so that flexibility is not a factor, the relation of the *ex ante* envelope technology and *ex post* total technology is analogous to that of the *ex ante* and *ex post* total cost curves of Figure 1(b): The optimal design will choose an *ex post* technology that will be operated at a point of "mutual tangency" of this technology and the *ex ante* envelope technology. We show below that this geometric property continues to hold in the general case where the firm's environment induces it to trade flexibility for efficiency. Note that the *ex ante* envelope technology contains explicit information about the tradeoff between flexibility and efficiency. We conclude that this technology is the appropriate generalization of the production structure underlying the classical *ex ante* total cost curves of Figures 1 and 2.

The competitive spot market prices faced by the firm in a state s are denoted by an N-vector $\tilde{\mathbf{p}}_s = (\tilde{p}_{1s},\ldots,\tilde{p}_{Ns})$. Then, $\tilde{\pi}_s = \tilde{\mathbf{p}}_s \cdot \mathbf{x}_s = \sum_{n=1}^{N} \tilde{p}_{ns} \cdot x_{ns}$

[3]A commodity which is to be delivered in some future date (or state) s may be traded in a *futures* market and/or in a *spot* market. Trade in a futures market occurs at the present time, when *ex ante* decisions are being made. Trade in a spot market occurs at the date at which the commodity is to be delivered. The price of a commodity in a futures market may be expressed in present currency units (the *forward* price), with trade interpreted as an exchange of present currency units for a contract to deliver the commodity at date s. It may also be expressed in currency units at date s (the *future* price), with trade interpreted as an exchange of contracts to deliver the commodity in state s and to deliver currency in state s. The commodity price in the spot market (the *spot* price) is denominated in units of currency at date s. In the presence of a full set of futures markets for the "variable" commodities, spot markets will be redundant as long as no agent gains "new" information in the interval between the openings of the future and spot markets. Then, spot prices will equal future prices, and will be related to forward prices by a discount factor determined in a competitive bond market. In the absence of formal future markets other than the bond market, this formulation can be assumed to continue to hold provided firms are hypothesized to have point expectations of spot prices, or to be risk neutral so that spot prices have the interpretation of certainty equivalents. However, a preferable model formulation in this case is to introduce firm expectations explicitly in the description of future states S, as in the case (c) above. To avoid ambiguities in interpretation, we assume hereafter that formal futures markets do exist for variable commodities, and that spot prices equal future prices.

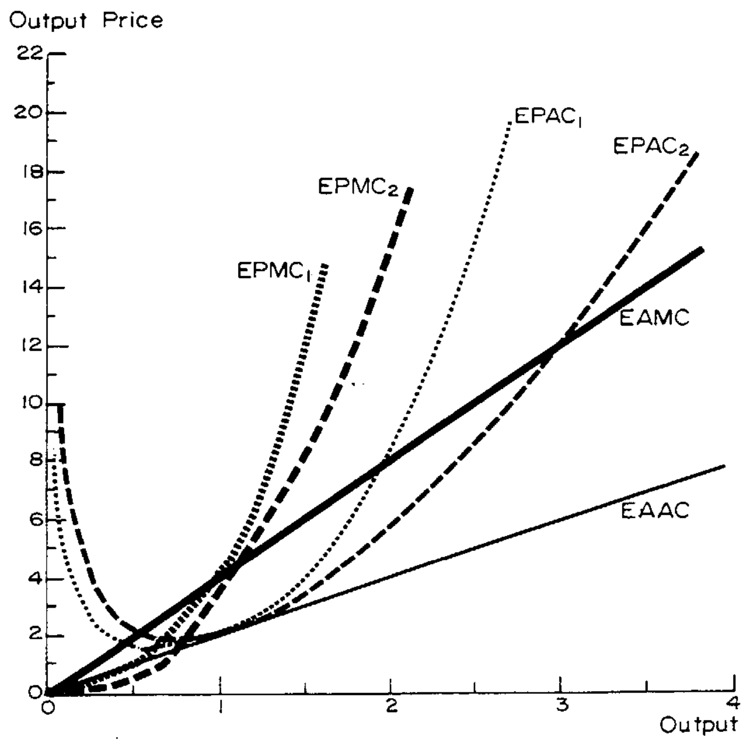

FIGURE 5(a)

is the variable profit in state s associated with the netput vector $\mathbf{x}_s$. We shall assume that the firm weighs (discounts) variable profit in state s by a factor δ_s, and has the objective for a given *ex post* variable technology $\mathbf{V}(\mathbf{b})$ of maximization of the "discounted sum" of variable profits over *ex post* production plans in $\mathbf{V}(\mathbf{b})$. For the model with intertemporal production and no uncertainty, δ_s is the discount rate from time s to reference time 0 established in a competitive bond market, $\tilde{\mathbf{p}}_s$ is a vector of spot prices, and the *ex post* objective of the firm is maximization of present value of variable profit. In the model with uncertainty and no intertemporal variation, δ_s is the probability that state s will occur, and the *ex post* objective of the firm is maximization of expected value of variable profit.[4]

The "discounted sum", or present value, of variable profit for an *ex post* production plan $\mathbf{x}$ can be written as the sum $\pi = \sum_{s \in \mathbf{S}} \delta_s \tilde{\mathbf{p}}_s \cdot \mathbf{x}_s =$

[4]In the model of intertemporal production $\delta_s \cdot \tilde{p}_{is}$ is a forward price and the forward prices are the variables with respect to which the firm maximizes present value. In the single-period uncertainty model, $\delta_s \cdot \tilde{p}_{is}$ is a certainty-equivalent price and the existence of competitive contingency bond markets implies that the firm will maximize expected value of profit, using certainty–equivalent prices as the exogenous variables.

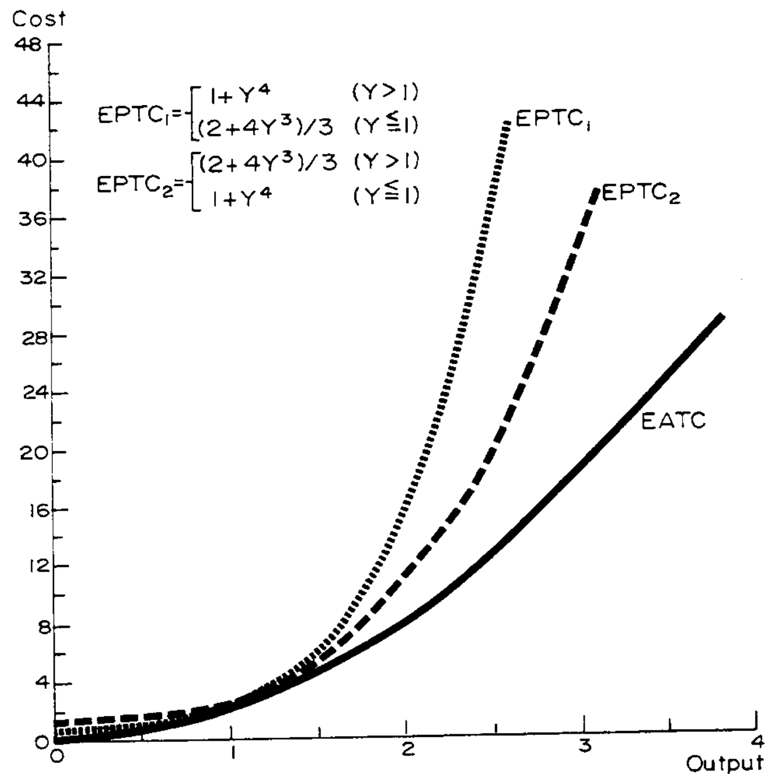

FIGURE 5(b)

$\sum_{s\in S} \delta_s \tilde{\pi}_s$ when **S** is a finite set, and as an integral $\pi = \int_S \delta_s \tilde{\mathbf{p}}_s \cdot \mathbf{x}_s \, d\mu(s)$ in the general case of a measure space $(\mathbf{S}, \mathcal{S}, \mu)$ of future states. Define an N-vector $\mathbf{p}_s = \delta_s \tilde{\mathbf{p}}_s$ of forward prices for state s and a *forward price vector* $\mathbf{p} = (\mathbf{p}_s : s \in \mathbf{S})$.[5] The present value of variable profit, hereafter termed *intertemporal variable profit*, can then be written in the case of finite **S** as the inner product of the vectors **p** and **x**, $\pi = \mathbf{p}\cdot\mathbf{x} = \sum_{s\in S} \mathbf{p}_s \cdot \mathbf{x}_s$. We shall carry over this inner product notation to the general case $\pi = \mathbf{p}\cdot\mathbf{x} \equiv \int_S \mathbf{p}_s \cdot \mathbf{x}_s \, d\mu(s)$.

We emphasize that the intertemporal variable profit $\pi = \mathbf{p}\cdot\mathbf{x}$ of an *ex post* production plan includes the present value of the quasi-rents accruing to the fixed inputs of structure and equipment. We can define the fixed cost of the firm by assuming that the capital inputs $\mathbf{K} = (K_1, \ldots, K_J)$ can be purchased in competitive markets at prices given by a

[5]In the following sections we shall use the language of intertemporal model since it is the most familiar one. From the preceding footnote it should be clear that only a slight change of language is required to render the analysis applicable to the other cases described in the paper.

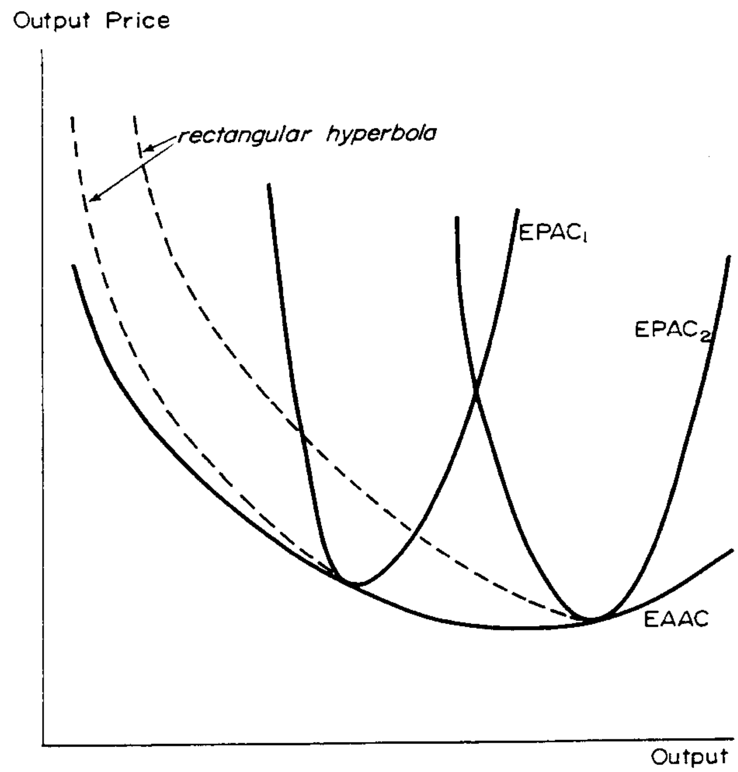

FIGURE 6

vector $\mathbf{r} = (r_1,...,r_J)$. We employ the convention that the signs of quantities and prices of these durable inputs are defined so that $K_j \leqq 0$ and $r_j \geqq 0$ for each j, and $\mathbf{r}\cdot\mathbf{K}$ is fixed cost with a negative sign. Then, the intertemporal total profit associated with a plant design $(\mathbf{a},\mathbf{K})$ and *ex post* production plan $\mathbf{x}$ is $\mathbf{p}\cdot\mathbf{x}+\mathbf{r}\cdot\mathbf{K}$.

The optimizing behavior of the competitive firm can now be summarized. Given an *ex post* variable technology $\mathbf{V}(\mathbf{b})$ with $\mathbf{b}\in\mathbf{B}$, maximization of intertemporal variable profit for a forward price vector $\mathbf{p}$ yields an *intertemporal variable profit function,*

$$\Pi(\mathbf{b},\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{x}|\mathbf{x}\in\mathbf{V}(\mathbf{b})\}. \tag{1}$$

The function Π is finite on a convex cone of prices for each design $\mathbf{b}$, and is a convex, conical (i.e., positively linear homogeneous), closed function of $\mathbf{p}$ on this cone (see Chapter II.2). Analogously, one may define an *ex post intertemporal total profit function,*

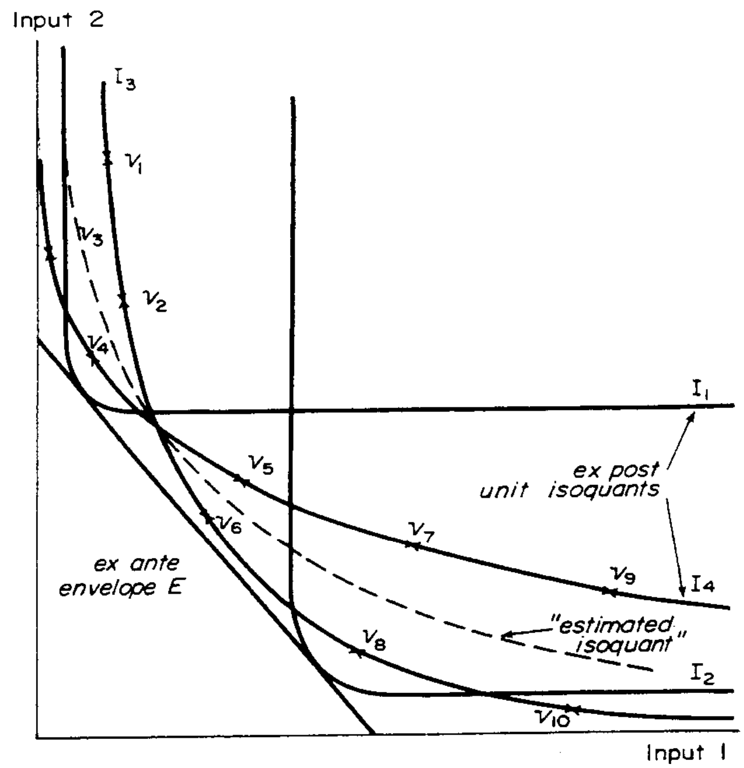

FIGURE 7

$$\phi(\mathbf{a},\mathbf{K},\mathbf{r},\mathbf{p}) = \Pi(\mathbf{a},\mathbf{K},\mathbf{p}) + \mathbf{r}\cdot\mathbf{K}. \tag{2}$$

Ex ante maximization of total profit over possible *ex ante* designs for a forward price vector **p** and durables price vector **r** yields an *ex ante envelope profit function*,

$$\begin{aligned}\Phi(\mathbf{r},\mathbf{p}) &= \sup\{\mathbf{r}\cdot\mathbf{K} + \mathbf{p}\cdot\mathbf{x} \mid (\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea}\}\\ &= \sup\{\mathbf{r}\cdot\mathbf{K} + \Pi(\mathbf{a},\mathbf{K},\mathbf{p}) \mid (\mathbf{a},\mathbf{K}) \in \mathbf{B}\}\\ &= \sup\{\phi(\mathbf{a},\mathbf{K},\mathbf{r},\mathbf{p}) \mid (\mathbf{a},\mathbf{K}) \in \mathbf{B}\}.\end{aligned} \tag{3}$$

In summary, for each possible *ex post* total technology and set of competitive markets, one can (normally) find an optimal *ex post* production plan and associated level of intertemporal total profit. The firm chooses *ex ante* a design that maximizes this profit, and then chooses *ex post* the actual optimal production plan given the *ex ante* design and the information available at the time the *ex post* decision is made.

4. Functional Forms for the *Ex Ante–Ex Post* Production Structure

The preceding section describes the two-stage (or two-level) decision procedure that is contained in our model of *ex ante–ex post* technological structures. In this section we will provide an intuitive and non-rigorous introduction to our procedure for generating quantitative *ex ante–ex post* technologies, and will illustrate (with the aid of examples) the use of this algorithm in the formation of econometric models. The mathematical justification of the procedure is contained in Section 5.

4.1. *The Algorithm*

The algorithm is an extension to the *ex ante–ex post* analysis of a result that is called "the derivative property of the restricted profit function" in Chapter II.2. This result states that the vector of partial derivatives of this function with respect to commodity prices, when it exists, equals a unique profit-maximizing netput bundle. That is,

$$\pi(\mathbf{q};\boldsymbol{\alpha}) = \sum_i q_i \cdot \hat{x}_i(\mathbf{q};\boldsymbol{\alpha}), \tag{4}$$

and

$$\frac{\partial \pi(\mathbf{q};\boldsymbol{\alpha})}{\partial q_i} = \hat{x}_i(\mathbf{q};\boldsymbol{\alpha}), \tag{5}$$

where $\pi(\mathbf{q};\boldsymbol{\alpha})$ is a restricted profit function, $\mathbf{q} = \{q_i\}$ is a vector of commodity prices, $\hat{\mathbf{x}} = \{\hat{x}_i\}$ is the profit-maximizing netput bundle, and $\boldsymbol{\alpha}$ is the vector of production parameters.

[Comparing (5) and the derivative of (4) with respect to q_k, one obtains the condition $\sum_i q_i \cdot \partial\hat{x}_i(\mathbf{q};\boldsymbol{\alpha})/\partial q_k = 0$.] The duality theorem for restricted profit functions (see McFadden's Chapter I.1) implies the existence of a unique technology $X(\boldsymbol{\alpha})$ satisfying $X(\boldsymbol{\alpha}) = \{\mathbf{x} | \mathbf{q}\cdot\mathbf{x} \leqq \pi(\mathbf{q},\boldsymbol{\alpha}) \text{ for all } \mathbf{q}\}$ and $\pi(\mathbf{q},\boldsymbol{\alpha}) = \sup\{\mathbf{q}\cdot\mathbf{x} | \mathbf{x} \in X(\boldsymbol{\alpha})\}$. Then, $\hat{\mathbf{x}}(\mathbf{q},\boldsymbol{\alpha}) \in \mathbf{X}(\boldsymbol{\alpha})$ and $\pi(\mathbf{q},\boldsymbol{\alpha}) \leqq \mathbf{q}\cdot\hat{\mathbf{x}}(\mathbf{q}',\boldsymbol{\alpha})$, with equality if $\mathbf{q} = \mathbf{q}'$. Using property (5), it is possible to generate single-level netput supply systems for econometric estimation of the underlying production parameters, starting from choice of an appropriate functional form for the restricted profit function (see, for example, Chapter II.2). The value of this approach is that it can provide closed functional forms for the netput supply system. The associated technology need not be described by a closed functional form. We emphasize that the algorithm does *not* provide a constructive process for obtaining profit functions from functional forms for the technology; it is, in fact, most useful in complex settings where it is impossible to obtain closed functional forms simultaneously for the profit function and the technology.

A similar algorithm holds for the two-stage, *ex ante–ex post* production structure. Suppose the *ex ante* envelope profit function can be written in the form

$$\Phi(\mathbf{r},\mathbf{p};\boldsymbol{\alpha}) = \sum_{i=1}^{L} Q_i(\mathbf{p})\hat{a}_i(\mathbf{r},\mathbf{p};\boldsymbol{\alpha}) + \sum_{j=1}^{J} r_j\hat{K}_j(\mathbf{r},\mathbf{p};\boldsymbol{\alpha}), \tag{6}$$

where $\mathbf{p}$ is the vector of forward prices, the $\hat{a}_i$ and $\hat{K}_j$ are the parameters, variable *ex ante* and fixed *ex post*, which specify the optimal *ex post* technology (or, equivalently, the optimal *ex ante* design), and $\boldsymbol{\alpha}$ is a vector of underlying *ex ante* parameters. The $Q_i(\mathbf{p})$ are functions of the vector of forward prices and play a role analogous to the prices q_i in the single-level formulation. The parameters of the optimal *ex post* technology satisfy

$$\partial\Phi/\partial Q_i = \hat{a}_i(\mathbf{r},\mathbf{p};\boldsymbol{\alpha}), \qquad \partial\Phi/\partial r_j = \hat{K}_j(\mathbf{r},\mathbf{p};\boldsymbol{\alpha}), \tag{7}$$

and the intertemporal profit-maximizing variable netputs satisfy

$$\begin{aligned}\partial\Phi/\partial p_{is} &= \sum_{k=1}^{L} (\partial\Phi/\partial Q_k)(\partial Q_k/\partial p_{is}) \\ &= \sum_{k=1}^{L} \hat{a}_k(\mathbf{r},\mathbf{p};\boldsymbol{\alpha})(\partial Q_k/\partial p_{is}) = x_{is}(\mathbf{r},\mathbf{p};\boldsymbol{\alpha}).\end{aligned} \tag{8}$$

(These relationships will be derived more rigorously in Section 5.) Writing Φ as a "nested" function $\Phi(\mathbf{r},\mathbf{p};\boldsymbol{\alpha}) = \psi(Q_1(\mathbf{p}),\ldots,Q_L(\mathbf{p}),r_1,\ldots,r_J;\boldsymbol{\alpha})$, and choosing appropriate convex conical closed functional forms for ψ and the Q_i, one can use equations (7) and (8) to generate estimable netput systems for the *ex ante–ex post* production model.

We will now proceed, by means of a series of examples, to show that a number of interesting cases can be analyzed using the above format.

Example 1: *Cobb–Douglas Production with Durable Capital*

Consider the classical model of a firm that has an *ex post* production function in each state s and produces a single output Y_s from a single variable input L_s and a single input K, fixed *ex post*. Then, the choice of K is the single *ex ante* design decision.

Assume the technology to be "separable across states" in the sense that the production possibilities in any state s are independent of operating decisions made for the remaining states. Let $Y_s = f(K,L_s;s)$ be a functional form defining this technology. A corresponding profit function, $\pi_s = \Pi_s(K,p_{Ys},p_{Ls})$, will specify optimal operation of the plant *ex post*, given forward prices p_{Ys}, p_{Ls} for the variable commodities in this state. One method of obtaining Π_s is to choose a functional form for the production function f and solve the *ex post* optimization problem explicitly. This procedure is satisfactory for some common production functions (e.g., Cobb–Douglas), but for others such as the C.E.S., an explicit closed-form solution for the *ex post* profit function is, in general, impossible. An alternative procedure is to specify a profit function directly, using duality theory to ensure that it is the solution to the optimization problem for some implicitly defined technology.[6] We illustrate this procedure for the Cobb–Douglas case:

The profit function

$$\pi_s = \Pi_s(K,p_{Ys},p_{Ls}) = A_{0s}K^{\alpha}p_{Ys}^{1+\beta}p_{Ls}^{-\beta} \tag{9}$$

is dual to the Cobb–Douglas production function

$$Y_s = (1+\beta)\beta^{-\beta/(1+\beta)}A_{0s}^{1/(1+\beta)}K^{\alpha/(1+\beta)}L_s^{\beta/(1+\beta)}, \tag{10}$$

where A_{0s} is an efficiency index for state s that incorporates all depreciation effects.

The *ex ante* design problem of the firm is to choose K to maximize

$$\pi = \int_S \Pi_s(K,p_{Ys},p_{Ls})\mathrm{d}\mu(s) - r\cdot K, \tag{11}$$

where r is the purchase price of capital. Now π can be written

$$\begin{aligned}\pi &= K^{\alpha}\int_S A_{0s}p_{Ys}^{1+\beta}p_{Ls}^{-\beta}\,\mathrm{d}\mu(s) - r\cdot K \\ &= a_1\cdot Q_1(\mathbf{p}) + r\cdot(-K),\end{aligned} \tag{12}$$

where $(a_1,-K)$ are the *ex post* (fixed) parameters contained in the set $\mathbf{B} = \{(a_1,-K)|a_1 = K^{\alpha},K \geqq 0\}$, and

[6]The conditions on Π_s for this construction are that it be a convex, positively linear homogeneous closed function of (p_{Ys},p_{Ls}). The specification that Y is an output and L is an input requires that Π_s be increasing in p_{Ys} and decreasing in p_{Ls}. The specification that K is an input (with a positive marginal product) requires that Π_s be increasing in K. The specification that the implicitly defined technology be convex (i.e., display generalized diminishing returns) requires that Π_s be concave in K.

$$\mathbf{p} = (\mathbf{p}_Y, \mathbf{p}_L) \quad \text{for} \quad \mathbf{p}_Y = \{p_{Ys} : s \in \mathbf{S}\}, \quad \mathbf{p}_L = \{p_{Ls} : s \in \mathbf{S}\},$$

$$Q_1(\mathbf{p}) = \int_{\mathbf{S}} A_{0s} p_{Ys}^{1+\beta} p_{Ls}^{-\beta} \, \mathrm{d}\mu(s).$$

Defining

$$\psi(Q_1, r) = \sup\{Q_1 a_1 + r(-K) | (a_1, -K) \in \mathbf{B}\}, \tag{13}$$

for $Q_1 > 0$, $r > 0$, and solving the maximization problem we have

$$\psi(Q_1, r) = \alpha^{\alpha/(1+\alpha)}(1-\alpha) r^{-\alpha/(1-\alpha)} Q_1^{1/(1-\alpha)} \tag{14}$$

$$= \hat{a}_1 Q_1(\mathbf{p}) + r(-\hat{K}), \tag{15}$$

where $\hat{a}_1$, $\hat{K}$ are the optimal *ex post* parameters. Then,

$$\frac{\partial \psi}{\partial Q_1} = \hat{a}_1 = \alpha^{\alpha/(1-\alpha)} Q_1^{\alpha/(1-\alpha)} r^{-\alpha/(1-\alpha)}$$

$$= \alpha^{\alpha/(1-\alpha)} \left[\int_{\mathbf{S}} A_{0s} p_{Ys}^{1+\beta} p_{Ls}^{-\beta} \, \mathrm{d}\mu(s) \right]^{\alpha/(1-\alpha)} r^{-\alpha/(1-\alpha)}, \tag{16}$$

$$\frac{\partial \psi}{\partial r} = -\hat{K} = -\alpha^{1/(1-\alpha)} Q_1^{1/(1-\alpha)} r^{-1/(1-\alpha)}$$

$$= -\alpha^{1/(1-\alpha)} \left[\int_{\mathbf{S}} A_{0s} p_{Ys}^{1+\beta} p_{Ls}^{-\beta} \, \mathrm{d}\mu(s) \right]^{1/(1-\alpha)} r^{-1/(1-\alpha)}. \tag{17}$$

Equations (16) and (17) specify the optimal *ex post* parameter vector $\hat{\mathbf{b}} = (\hat{a}_1, -\hat{K})$ in terms of the price vector $\mathbf{p}$ and the underlying *ex ante* parameters α, β, and $A_{0s}, s \in \mathbf{S}$. This system of equations is nonlinear in the underlying parameters and difficult to implement empirically. However, it illustrates the relationship between ψ and the *ex post* parameters that will be used in a later example to obtain a more tractable system.

Example 2: *Activity Analysis with Capacity Constraints*

Suppose a firm has available *ex ante* a finite set $\mathbf{T}^{ea}$ of possible activities $(K^j, \mathbf{x}^j)$, $j = 1,\ldots,J$, with K^j specifying the capital input per unity of capacity required by activity j, with a negative sign, and $\mathbf{x}^j$ specifying the *ex post* production plan (per-unit capacity) for activity j. Let $\mathbf{a} = (a_1,\ldots,a_J)$ be a design vector specifying the capacities in an *ex post* technology; i.e., activity j can be operated *ex post* at any intensity level up to a_j. A set $\mathbf{B}$ of possible vectors $\mathbf{a} = (a_1,\ldots,a_J)$ specifies the designs that are available *ex ante*. In the "putty–putty" case, positive capacity can be provided *ex post* for all activities (i.e., $\mathbf{B}$ is a rectangle). In the "putty–clay" case, the activities are mutually exclusive *ex post*, and, at most, one capacity can be positive (i.e., **B** is the set of vertices of a simplex). The *ex post* technology is

$$V(\mathbf{a}) = \left\{ \sum_{j=1}^{J} (a_{sj} x_s^j : s \in \mathbf{S}) | 0 \leqq a_{sj} \leqq a_j \right\},$$

where a_j is the chosen capacity of activity j, fixed *ex post* and a_{sj} is an *ex post* intensity of operation. One can also write

$$V(\mathbf{a}) = \sum_{j=1}^{J} a_j V^j, \tag{18}$$

where $V^j = \{\mathbf{x} | x = \gamma_s^j \cdot x_s^j, 0 \leqq \gamma_s^j \leqq 1\}$, implying $a_{sj} = \gamma_s^j a_j$. The intertemporal variable profit function is then

$$\pi(\mathbf{a},\mathbf{p}) = \sum_{j=1}^{J} a_j \pi^j(\mathbf{p}), \tag{19}$$

where

$$\begin{aligned} \pi^j(\mathbf{p}) &= \int_S \max(p_s \cdot x_s^j, 0) \mathrm{d}\mu(s) \\ &= \sup\{\mathbf{p}\cdot\mathbf{x} | \mathbf{x} \in V^j\}. \end{aligned} \tag{20}$$

The intertemporal total profit function then equals $\sum_{j=1}^{J} a_j[\pi^j(\mathbf{p}) + r \cdot K^j]$, and the *ex ante* envelope profit function satisfies

$$\Phi(r,\mathbf{p}) = \sup_{\mathbf{a} \in \mathbf{B}} \sum_{j=1}^{J} a_j [\pi^j(\mathbf{p}) + r \cdot K^j]. \tag{21}$$

We can define a profit function for the design set **B**,

$$\psi(Q_1,\ldots,Q_j) = \sup_{\mathbf{a} \in \mathbf{B}} \sum_{j=1}^{J} a_j \cdot Q_j, \tag{22}$$

where $Q_j = \pi^j(\mathbf{p}) + r \cdot K^j$.

Then, the *ex ante* envelope profit function has the form

$$\Phi = \psi(Q_1,\ldots,Q_J) = \sum_{j=1}^{J} \theta_j \cdot Q_j, \tag{23}$$

where θ_j satisfies the maximization problem. For example, in a putty–clay case, where the maximum designed capacity of any activity is unity,

$$\Phi = \psi(Q_1,\ldots,Q_J) = \sum_j \theta_j Q_j = \max_j Q_j = \max_j(\pi^j(\mathbf{p}) + r \cdot K^j). \tag{24}$$

Example 3: *Base Load Versus Peaking Capacity in an Electricity Generating System*

An electric utility is required to meet an output demand that has a known expected daily cycle (normalized in this example so that anticipated peak demand is equal to one), and some random variation. Let t denote time of day, scaled so that $0 \leqq t \leqq 1$, and $l(t)$ denote instantaneous output demand as a fraction of capacity (termed the system load factor) at time t. Figure 8 illustrates a typical expected load curve. The total daily output of the system, expressed as a fraction of maximum daily output, or average load factor, equals $\int_0^1 l(t)dt$, the area under the load curve in Figure 8.

For any load factor l, define $f(l)$ to be the fraction of the day for which the system load factor $l(t)$ is at least l, as illustrated in Figure 8. Clearly, f is a non-increasing function of l, with $f(0)=1$ and $f(1) \geqq 0$. Further, the average load factor of the system equals $\int_0^1 f(l)dl$. A typical "time-at-load" function f is illustrated in Figure 9(a).

The utility can provide capacity by constructing base-load or peaking plants or by contracting to purchase power from an electricity grid. The

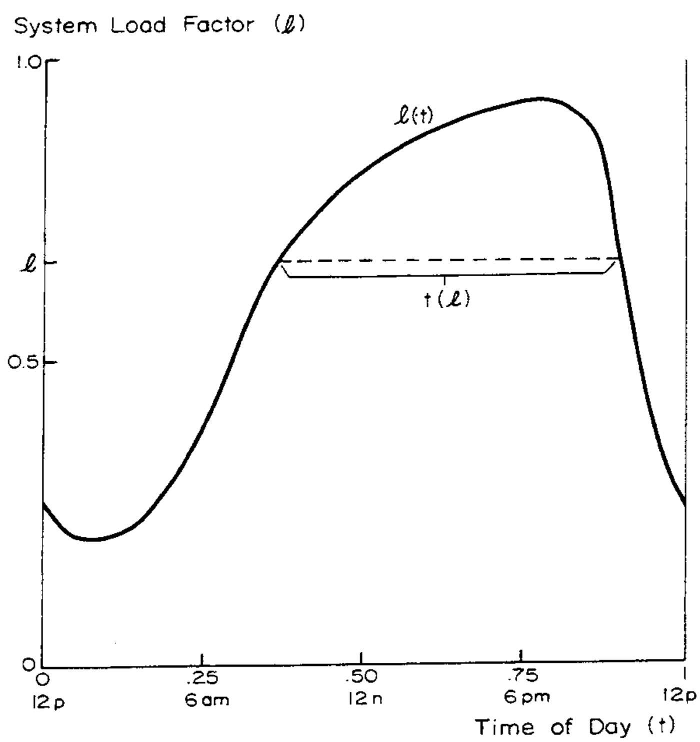

FIGURE 8

following notation is employed for the costs of these production modes:

	Base-load plant	Peaking plant	Grid purchase
Initial capital cost per unit of capacity ($/kw)	k_b	k_p	0
Present value of operating cost ($/kw)	c_b	c_p	c_g

The *ex ante* parameters of the supply process are subsumed in the unit costs, for differences in these costs reflect basic underlying differences in the possible methods of "producing" electricity.

For simplicity, we assume that these costs are constant, independent of plant scale and of the load curve. We also assume that the utility has no discretionary control over sales to the electricity grid.

Of the three production modes, base-load plant has the highest capital cost and the lowest total cost of providing continuous capacity output, while grid purchase is the most expensive mode for providing continuous output: $k_b > k_p$ and $k_b + c_b < k_p + c_p < c_g$. Suppose that *ex post*, the utility has a proportion α of its system capacity in base-load plant and a proportion β in combined base-load and peaking plant. In optimal operation, demand will be met first by base-load plant, second by peaking plant (for load factors above α), and last by purchased power (for load factors above β). Present value of total cost per unit of system capacity then equals[7]

$$C = \alpha k_b + (\beta - \alpha)k_p + c_p \int_0^\alpha f(l)\mathrm{d}l \mp c_p \int_\alpha^\beta f(l)\mathrm{d}l + c_g \int_\beta^1 f(l)\mathrm{d}l. \tag{25}$$

[7]More realistic assumptions on unit costs would be that there are some increasing returns to plant scale, particularly in small spatially concentrated utilities, and that marginal future operating costs for base-load and peaking plants depend on plant load factors. (There are generally decreasing marginal costs with load in a given unit, but increasing marginal costs in the system as successively less efficient units are operated at capacity.) The construction of the formula (25) is illustrated by derivation of the variable cost of base-load plant operation. Let $c_b(\lambda)$ equal the marginal cost of operating a unit capacity base-load plant at load factor λ. Then, $\gamma(\lambda) = \int_0^\lambda c_b(\lambda')\mathrm{d}\lambda'$ is the total variable cost of operating this plant at load factor λ. Suppose the utility has a system load curve $l(t)$, a proportion α of base-load capacity, and the policy of operating base-load capacity first. (Provided the marginal cost of peaking-plant operation exceeds that of base-load plant operation at any respective plant-load factors, this policy is always optimal.) Then the base-load plant is operated with load curve $\lambda(t) = \min((l(t)/\alpha), 1)$. Retaining the assump-

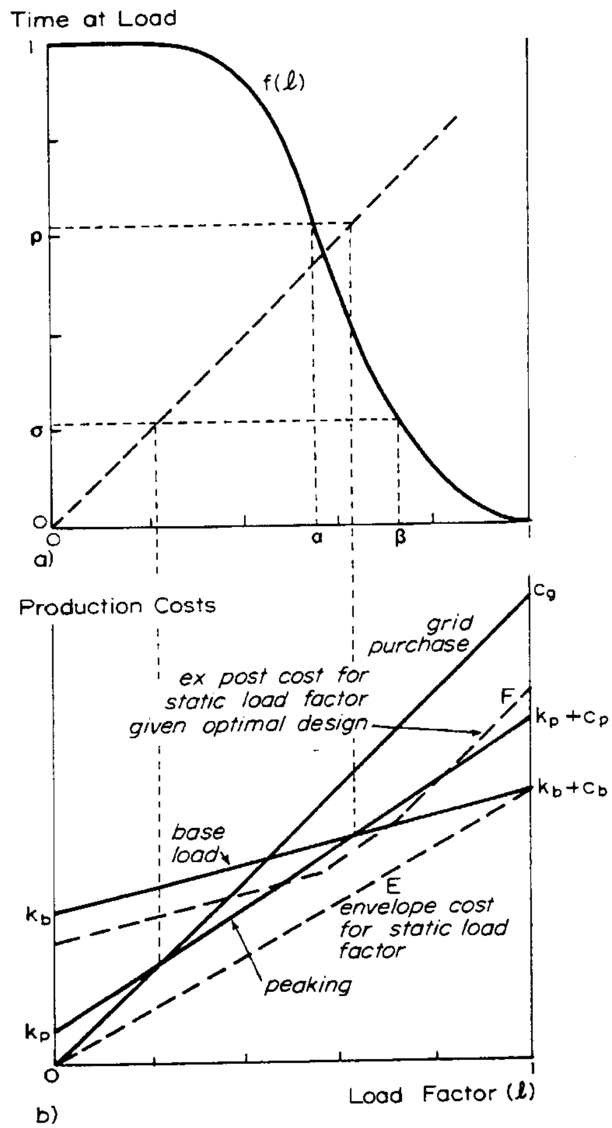

FIGURE 9

tion that marginal cost is independent of plant scale, the total variable cost of operating the base-load plant at load factor $\lambda(t)$, divided by system capacity, equals $\alpha\gamma(\lambda(t))$. Hence, the variable cost of base-load plant operation is

$$\int_0^1 \alpha\gamma(\lambda(t))dt = \alpha \int_{t=0}^{1} \int_{\lambda=0}^{\min((l(t)/\alpha),\,1)} c_b(\lambda)d\lambda\, dt = \alpha \int_{\lambda=0}^{1} \int_{t=t_1(\alpha\lambda)}^{t_2(\alpha\lambda)} dt\, c_b(\lambda)d\lambda$$
$$= \alpha \int_{\lambda=0}^{1} f(\alpha\lambda)c_b(\lambda)d\lambda = \int_0^{\alpha} f(l)c_b(l/\alpha)dl,$$

where the second equality is obtained by interchanging the order of integration (see Figure 8), and the third equality follows from the definition of $f(l)$. When $c_b(l/\alpha)$ is constant, the corresponding term in (25) is obtained.

Define $\rho = (k_b - k_p)/(c_p - c_b)$ and $\sigma = k_p/(c_g - c_p)$. Note from Figure 9(b) that ρ is the load factor at which unit capacity base-load and peaking plants would yield the same total cost. Similarly, σ is the load factor at which purchased power and a unit capacity peaking plant yield the same total cost. Consider, now, the *ex ante* minimization of C in the design parameters α and β. Note that $\partial C/\partial\alpha = (c_p - c_b)(\rho - f(\alpha))$ and $\partial C/\partial\beta = (c_g - c_p)(\sigma - f(\beta))$. We confine our attention to the case $\sigma \leqq \rho$ illustrated in Figure 9(b) in which there is a range of load factors for which a unit capacity peaking plant will yield lower total cost than either a unit capacity base-load plant or purchased power. Then the *ex ante* minimum occurs for α and β satisfying $f(\alpha) = \rho$, $f(\beta) = \sigma$. The determination of these quantities is illustrated in Figure 9(a).[8]

The total cost function (25) is nonlinear in the design parameters α, β. However, we can reparameterize *ex ante* design possibilities to make total cost linear in design parameters. Define **B** to be the set of vectors $\mathbf{a} = (a_1,\dots,a_5) = (\alpha, \beta - \alpha, \int_0^\alpha f(l)\mathrm{d}l, \int_\alpha^\beta f(l)\mathrm{d}l, \int_\beta^1 f(l)\mathrm{d}l)$ for $0 \leqq \alpha \leqq \beta \leqq 1$, and define $\psi(Q_1,\dots,Q_5) = \min\{\sum_i Q_i a_i | \mathbf{a} \in \mathbf{B}\}$. Then, $C = a_1k_b + a_2k_p + a_3c_b + a_4c_p + a_5c_g$, and the *ex ante* envelope cost function equals $\psi(k_b,k_p,c_b,c_p,c_g)$. Further, ψ has the property that its derivative with respect to Q_i, evaluated at (k_b,k_p,c_b,c_p,c_g), equals the optimal value of the corresponding design parameter $\hat{a}_i$.

It is of interest to compare the cost curves generated by this concrete model with the classic curves in Figures 1 and 2. A system with unit capacity, all base-load, has the total cost curve $k_b + c_b l$ in Figure 9(b). A least-cost unit capacity system to produce an output θ uniform in time has base-load capacity θ, zero peaking capacity, and purchased power capacity $1 - \theta$. The line E is the envelope of such least-cost unit capacity systems for various θ. The curve F in this figure is the total cost curve for the optimal system with the "time-at-load" curve illustrated in Figure 9(a). Note that there is no single output uniform in time for which this system remains optimal. Note, further, that a shift in the load curve that increases variability while keeping total output constant will generally decrease the proportion of optimal base-load capacity and increase the proportion of peaking capacity.

The reader may find it useful to verify these conclusions for an example. Suppose the utility has the system "time-at-load" function illustrated in Figure 10(b), which yields an average load u and a variance

[8]In the case $\sigma > \rho$, no peaking capacity will be constructed ($\alpha = \beta$), and optimal α satisfies $f(\alpha) = k_b/(c_g - c_b)$.

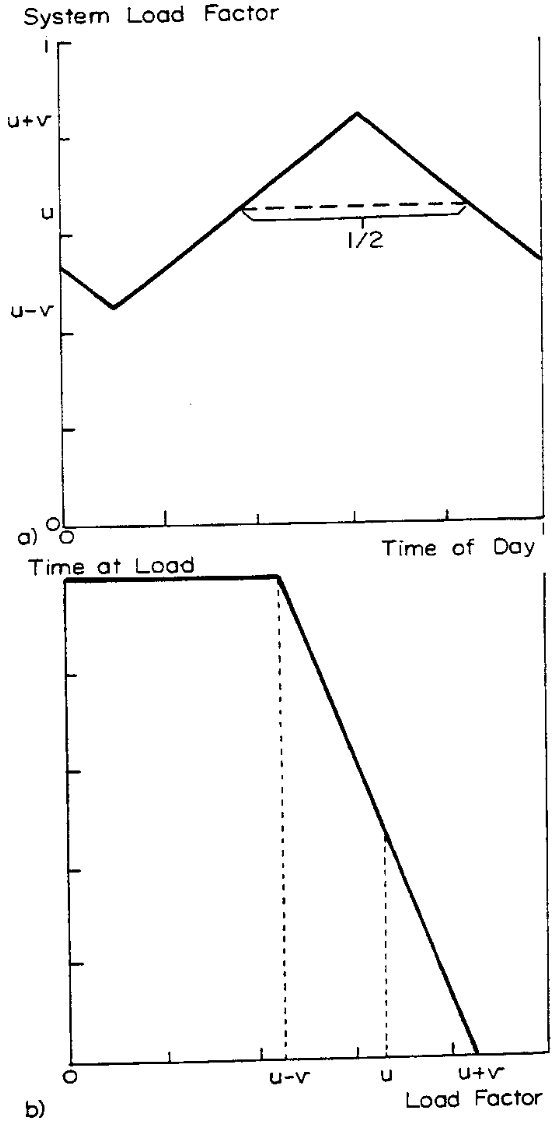

FIGURE 10

in load $v^2/3$. Minimization of $\Sigma\, Q_i a_i$ over $\mathbf{a} \in \mathbf{B}$ yields

$$\begin{aligned} \psi(Q_1,\ldots,Q_5) = {} & (Q_1 + Q_3)u - Q_2^2/(Q_5 - Q_4)], \\ & + v[Q_1 - (Q_1 - Q_2)^2/(Q_4 - Q_3) \end{aligned} \tag{26}$$

in the case $(Q_1 - Q_2)/(Q_4 - Q_3) > Q_2/(Q_5 - Q_4)$, and an optimal *ex ante* design $\hat{\alpha} = u + v - 2v(k_b - k_p)/(c_p - c_b)$ and $\hat{\beta} = u + v - 2vk_p/(c_g - c_p)$. Note that an increase in the variance of the load with average load fixed will always increase peaking capacity. Under normal conditions, unit costs satisfy $\sigma < \rho < 1/2$, implying that base load capacity will increase and grid purchases will decrease as load variance rises.

Example 4: A Nested Ex Ante–Ex Post Functional Form for the Two-Level Technology

The model presented below was developed by the authors as an extension of Diewert's generalized Leontief cost function. It has been analyzed extensively in Fuss (1970, 1977b), and its parameters estimated by Fuss in Chapter IV.4, using electricity generation data, for the case of intertemporal variation with no uncertainty. We summarize the model's basic characteristics. The reader is referred to the cited works for a more detailed description.

For this example the "*a,b*" notation is reversed from that in the rest of the paper in order to retain consistency with the cited literature. Thus, for this example *only*, **b** refers to a vector of *ex post* fixed design variables (rather than **a**), and **a** refers to a vector of characteristics of the underlying technology.

Suppose a producer expects to use n variable factors to produce one unit of output in each future time period t and state of nature u_t in period t, where factor prices vary with u_t in each period and the u_t in different periods are statistically independent. A future state is then a vector $\mathbf{s} = (t,u_t : t = 1,2,...)$. Suppose the *ex post* variable cost function in state s is Diewert's second-order approximation to an arbitrary unit cost function,

$$C_t\left(\tilde{p}_{1tu_t},\dots,\tilde{p}_{ntu_t}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\left(\tilde{p}_{itu_t}\tilde{p}_{jtu_t}\right)^{1/2}, \tag{27}$$

where b_{ij} is an *ex post* parameter and $\tilde{p}_{itu_t}$ is a future price. The present value of expected cost is

$$C = \sum_t \delta^t E_{u_t}(C_t) = \sum_i \sum_j b_{ij}Q_{ij}, \tag{28}$$

where

$$Q_{ij} = \sum_t \delta^t E_{u_t}\left(\tilde{p}_{itu_t}\tilde{p}_{jtu_t}\right)^{1/2} = \sum_t E_{u_t}\left(p_{itu_t}p_{jtu_t}\right)^{1/2},$$

and p_{itu_t} is a forward price. Note that C is linear in the *ex post* parameters since Q_{ij} is a function of prices alone and is analogous to the terms Q_i appearing in the previous examples. It has been shown in Fuss (1977b) that if we specify as a functional form for the *ex post* parameters

$$b_{ij} = \left(\frac{1}{d_{ij}}\right) \sum_{k,l=1,\ldots,n} (d_{kl}d_{ij})^{1/2} a_{ijkl}, \tag{29}$$

where d_{ij}, d_{kl} are arbitrary non-negative "design" variables and a_{ijkl} are fixed *ex ante* parameters, then C is minimized when

$$\hat{b}_{ij} = \frac{1}{Q_{ij}} \sum_{k} \sum_{l} (Q_{kl}Q_{ij})^{1/2} a_{ijkl}. \tag{30}$$

Note that the optimal *ex post* parameters are linear in the *ex ante* parameters.

We can now describe this structure in the format developed in the previous examples. The *ex ante* envelope cost function is

$$C(\{Q_{ij}\}|i,j = 1,\ldots,n) = \min_{\mathbf{B}} \left\{ \sum_{i,j} Q_{ij} \cdot b_{ij} \right\}, \tag{31}$$

where the design parameters are contained in the set

$$\mathbf{B} = \left[\left\{ b_{ij} = \frac{1}{d_{ij}} \sum_{k,l} (d_{kl}d_{ij})^{1/2} a_{ijkl} \right\} \middle| d_{ij}, d_{kl} \geqq 0 \right].$$

The solution to this minimization problem is

$$\begin{aligned} C(\{Q_{ij}\}) &= \sum_{ij} Q_{ij} \cdot \hat{b}_{ij} \\ &= \sum_{ij} \sum_{kl} (Q_{ij}Q_{kl})^{1/2} a_{ijkl}, \end{aligned} \tag{32}$$

where $\hat{b}_{ij}$ is given by (30). Note that C is linear in the *ex ante* parameters and is therefore amenable to estimation using standard regression techniques.

Using the theory outlined at the beginning of this section, we could have begun with the *ex ante* envelope cost function $C = \sum_{i,j,k,l} (Q_{ij}Q_{kl})^{1/2} a_{ijkl}$ and derived the optimal *ex post* parameters and netput bundles. That is,

$$\hat{b}_{ij} = \frac{\partial C}{\partial Q_{ij}} = \frac{1}{Q_{ij}} \sum_{kl} (Q_{kl}Q_{ij})^{1/2} a_{ijkl},$$

and

$$\hat{x}_{itu_t} = \frac{\partial C}{\partial [p_{itu_t}]} = \sum_j \frac{\partial C}{Q_{ij}} \cdot \frac{\partial Q_{ij}}{\partial [p_{itu_t}]} = \sum_j \hat{b}_{ij} \cdot \frac{\partial Q_{ij}}{\partial [p_{itu_t}]}.$$

If the p_{it} are assumed known with certainty, then

$$\hat{x}_{it} = \sum_j \hat{b}_{ij} \left(\frac{p_{jt}}{p_{it}}\right)^{1/2}, \tag{33}$$

since

$$\frac{\partial Q_{ij}}{\partial p_{it}} = \left(\frac{p_{jt}}{p_{it}}\right)^{1/2}.$$

Then, using (30),

$$\hat{x}_{it} = \sum_{j,k,l} \left[\left(\frac{Q_{kl}}{Q_{ij}}\right)^{1/2} \left(\frac{p_{jt}}{p_{it}}\right)^{1/2}\right] a_{ijkl}, \tag{34}$$

which is the expression for the *ex post* optimal inputs found in Fuss (1977b).

The structure above does not identify explicitly inputs of capital equipment whose levels are set *ex ante*. With some added notation, this can be done as follows: Suppose, of the n "variable" factors above, the first J are identified as inputs of capital equipment. Define the set of indices $N = \{(i,j) | i = j \leqq J \text{ or } J < i,j \leqq n\}$. Define $Q_{jj} = r$, the price of capital good j, for $j = 1,\dots,J$, and define Q_{ij} as above for $i,j > J$. Define the *ex ante* envelope cost function

$$C = \sum_{ij\in N} \sum_{kl\in N} (Q_{ij}Q_{kl})^{1/2} a_{ijkl}.$$

Then, the optimal *ex post* parameters satisfy $\partial C/\partial Q_{jj} = \hat{K}_j$ for $j = 1,\dots,J$; $\partial C/\partial Q_{ij} = \hat{b}_{ij}$ for $i,j > J$; and $\partial C/\partial p_{itu_t} = \hat{x}_{itu_t}$ for $i > J$.

We shall now illustrate that this functional form can be used to specify the flexibility–efficiency tradeoff by considering a simplified two-factor example with one operating period.

Suppose the forward prices p_{1u}, p_{2u} satisfy $E_u(p_{iu}^{1/2}) = \mu$, $Q_{12} = E_u(p_{1u}p_{2u})^{1/2} = \mu^2$, and $Q_{ii} = E_u(p_{iu}) = \mu^2 + \sigma^2 \equiv \mu^2\beta^2$. Then, the two price series are uncorrelated, and each has a variance $\sigma^2 = \mu^2(\beta^2 - 1)$. (Thus, $\beta \geqq 1$ is an increasing index of price variability.) For further simplicity, assume the underlying *ex ante* parameters a_{ijkl} take the values in the following table:

		Index ij			
		11	12	21	22
Index kl	11	0	1/2	1/2	A
	12	1/2	A	A	1/2
	21	1/2	A	A	1/2
	22	A	1/2	1/2	0

where A is a non-negative scalar. From equation (30), $\hat{b}_{11} = \hat{b}_{22} = A + \beta^{-1}$ and $\hat{b}_{12} = \hat{b}_{21} = A + \beta$. For a given vector of forward prices (p_{1u}, p_{2u}), the input demands are

$$\hat{x}_{1u} = \hat{b}_{11} + \hat{b}_{12}(p_{2u}/p_{1u})^{1/2},$$
$$\hat{x}_{2u} = \hat{b}_{22} + \hat{b}_{21}(p_{1u}/p_{2u})^{1/2}. \tag{35}$$

Evaluated at the "mean" price vector $(p_{1s}, p_{2s}) = (\mu, \mu)$,

$$\hat{x}_{1u} = \hat{x}_{2u} = 2A + \beta^{-1} + \beta. \tag{36}$$

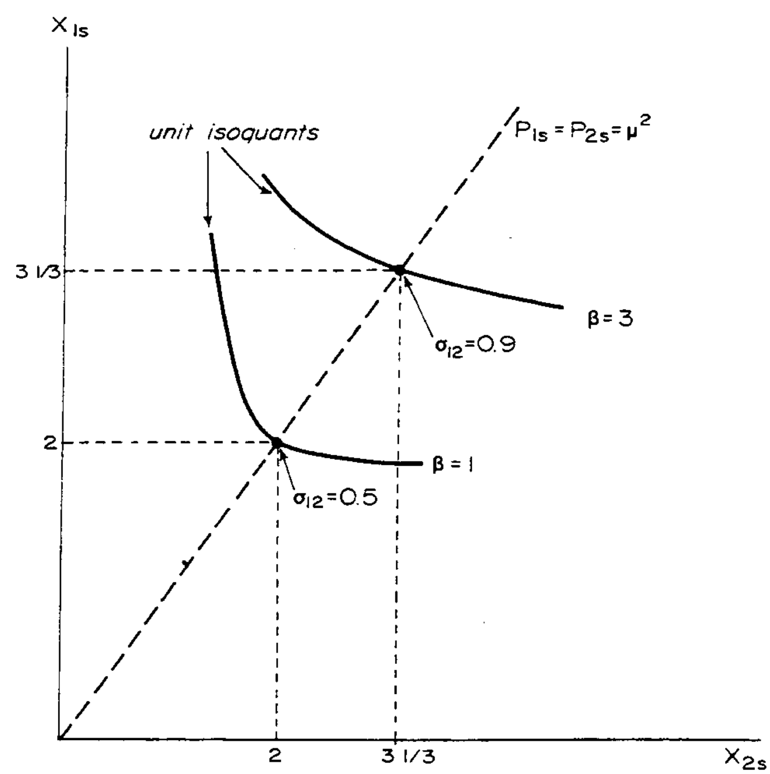

FIGURE 11

The *ex post* elasticity of factor substitution between inputs 1 and 2 is given by

$$\mathrm{ES} = (p_{1u}\hat{x}_{1u} + p_{2u}\hat{x}_{2u})(\partial \hat{x}_{1u}/\partial p_{2u})/\hat{x}_{1u}\hat{x}_{2u}. \tag{37}$$

Evaluated at $(p_1,p_2) = (E_u p_{1u}, E_u p_{2u}) = (\mu^2\beta^2, \mu^2\beta^2)$, this elasticity is

$$\mathrm{ES} = (A+\beta)/(2A+\beta+\beta^{-1}). \tag{38}$$

As $\beta \geqq 1$ is increased (price variability rises), ES increases (flexibility rises) and $\hat{x}_{1u}$ evaluated at the price vector $(E_u p_{1u}, E_u p_{2u})$ also increases (static efficiency falls). Figure 11 illustrates unit isoquants of the *ex post* technology for levels of β in the case $A = 0$.

Define as an index of static efficiency the ratio e of the input level (36) at $\beta = 1$ to this input level at the value of β corresponding to a given level of price variability. Figure 12 illustrates the tradeoff between *ex post* flexibility measured by the elasticity of substitution and this index of static efficiency. Note that as A increases, the possibility of substituting flexibility for efficiency falls.

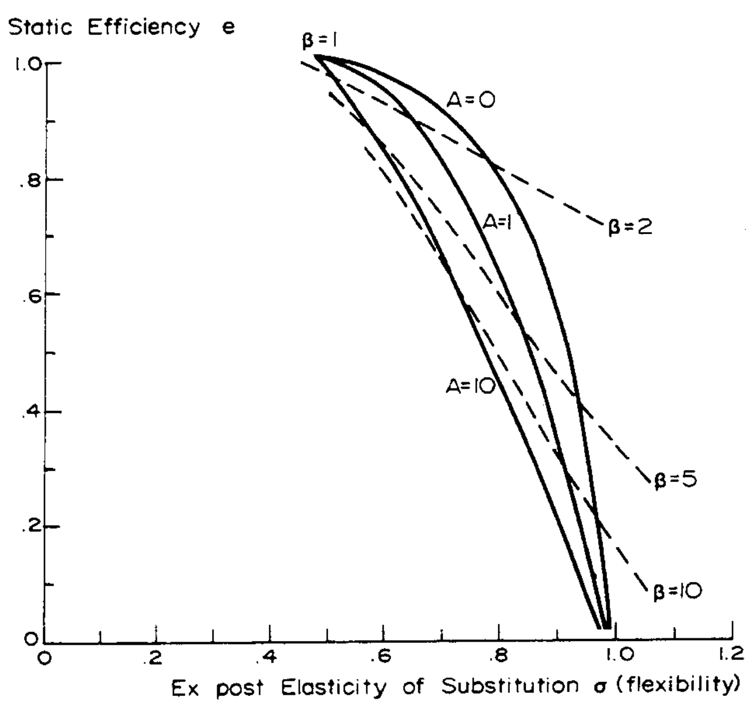

FIGURE 12

5. Derivation of the Two-Level Structure of Technology

In this section we use the model of an *ex ante–ex post* technology described in Section 3 and develop analytically the algorithm for generating econometric netput supply systems illustrated in Section 4. These examples have the common feature that *ex ante* design possibilities can be parameterized so that the *ex post* intertemporal total profit function is *linear* in the *ex ante* design variables (*ex post* parameters). It is this property and the duality of technologies and profit functions that we exploit to develop the desired algorithm. We define, on one hand, a family of *ex ante–ex post* technologies with the linear structure above and, on the other hand, a family of "nested" profit functions. The operations of profit maximization and of construction of the set of production plans consistent with profit maximization are shown to define one-to-one mutually inverse mappings between these families. Then, each "nested" profit function characterizes some *ex ante–ex post* technology. The algorithm is then to choose an appropriate functional form for a nested-profit function and use the derivative property to obtain expressions for the optimal *ex ante* design and optimal *ex post* netput supply vector. Properties of the associated technology are obtained implicitly from the profit function, using the duality mappings. We consider, first, the case in which the set of future states and the vector of *ex ante* design variables are finite.

5.1. *The Finite Case*

Suppose there are a finite number of future states, indexed $s = 1,\dots,S$. Suppose that an *ex ante* plant design $\mathbf{b} = (\mathbf{a},\mathbf{K})$ is composed of finite vectors $\mathbf{a} = (a_1,\dots,a_L)$ and $\mathbf{K} = (K_1,\dots,K_J)$. A *technological structure* is defined by an *ex ante* envelope technology $\mathbf{T}^{ea}$, a non-empty subset of $\mathbf{E}^{L+J+N\cdot S}$, and the following associated sets: the set of *ex ante* designs $\mathbf{B} = \{(\mathbf{a},\mathbf{K}) \in \mathbf{E}^{L+J} | \exists \mathbf{x} \ni (\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea}\}$, the *ex post* variable technology $\mathbf{V}(\mathbf{a},\mathbf{K}) = \{\mathbf{x} \in \mathbf{E}^{N\cdot S} | (\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea}\}$ defined for $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$, an *ex ante* netput possibility set $\mathbf{W} = \{(\mathbf{K},\mathbf{x}) \in \mathbf{E}^{J+N\cdot S} | \exists \mathbf{a} \ni (\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea}\}$, and the *normal cone* of $\mathbf{W}$, the set $\mathbf{F} \subseteq \mathbf{E}^{J+N\cdot S}$ of normals to hyperplanes whose lower half-spaces contain $\mathbf{W}$; i.e., $\mathbf{F}$ is the set of $(\mathbf{r},\mathbf{p}) \in \mathbf{E}^{J+N\cdot S}$ such that $\mathbf{r}\cdot\mathbf{K} + \mathbf{p}\cdot\mathbf{x}$ is bounded above for $(\mathbf{K},\mathbf{x}) \in \mathbf{W}$.

We term a technological structure *regular* if the following conditions hold: (1) the sets $\mathbf{B}$, $\mathbf{W}$, and $\mathbf{V}(\mathbf{a},\mathbf{K})$ for $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$ are closed, (2) capital inputs are non-positive and exhibit free disposal; i.e., $\mathbf{K} \leqq \mathbf{0}$ for all $(\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea}$, and $(\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea}$, $\mathbf{K}' \leqq \mathbf{K}$ implies $(\mathbf{a},\mathbf{K}',\mathbf{x}) \in \mathbf{T}^{ea}$, and (3) the normal cone $\mathbf{F}$ has a non-empty interior (denoted by $\mathbf{F}_0$). Note that $\mathbf{F}$ is the set of prices that yield finite maximum profits in the *ex ante* envelope technology. Condition (2), above, states our accounting convention for capital inputs. The free disposal assumption implies that r_j is non-negative for $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$, with $r_j K_j \leqq 0$ for all $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$. The condition that $\mathbf{F}_0$ be non-empty holds if and only if the set $\mathbf{W}$ is semibounded (see Chapter I.1). A sufficient condition for $\mathbf{F}_0$ non-empty is that there be some commodities that are essential inputs to production and cannot themselves be produced in the technology. It is *not* necessary for $\mathbf{W}$ to be convex in order to have $\mathbf{F}_0$ non-empty; however, $\mathbf{W}$ cannot exhibit indefinitely increasing returns to the extent that "average products" are unbounded with unbounded scale. When $\mathbf{T}^{ea}$ is convex, then $\mathbf{W}$, $\mathbf{B}$, and $\mathbf{V}(\mathbf{a},\mathbf{K})$ are convex and we say the technological structure is *convex*.

Suppose the technological structure is such that the *ex post* variable technology is linear in the parameter vector $\mathbf{a}$ and not explicitly dependent on $\mathbf{K}$. (An implicit dependence on $\mathbf{K}$ results from the relation of $\mathbf{a}$ and $\mathbf{K}$ in the set $\mathbf{B}$.) Then we can write

$$\mathbf{V}(\mathbf{a},\mathbf{K}) = \sum_{l=1}^{L} a_l \mathbf{V}^l = \left\{ \sum_{l=1}^{L} a_l \mathbf{x}^l \middle| \mathbf{x}^l \in \mathbf{V}^l \right\}, \tag{39}$$

where the $\mathbf{V}^l$ are non-empty subsets of $\mathbf{E}^{N \cdot S}$. We term the technological structure *design linear* if it satisfies equation (39) and if for each $l = 1,\ldots,L$, $\mathbf{V}^l$ is a closed set and either a_l is non-negative for all $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$ or $\mathbf{V}^l$ is a singleton.

A *regular profit structure* is defined by: (1) a convex cone $\mathbf{F}$ of price vectors $(\mathbf{r},\mathbf{p}) \in \mathbf{E}^{J+N \cdot S}$ such that its interior $\mathbf{F}_0$ is non-empty and r_j is non-negative for all $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$, $j = 1,\ldots,J$; (2) a non-empty closed set $\mathbf{B}$ of *ex ante* design variables $(\mathbf{a},\mathbf{K}) \in \mathbf{E}^{L+J}$ such that $\mathbf{K} \leqq \mathbf{0}$ for $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$; (3) a convex conical closed function of $\mathbf{p}$, $\Pi(\mathbf{a},\mathbf{K},\mathbf{p})$, defined for $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$ and $\mathbf{p}$ in the set $\mathbf{F}^p = \{\mathbf{p} \in \mathbf{E}^{N \cdot S} | \exists \mathbf{r} \ni (\mathbf{r},\mathbf{p}) \in \mathbf{F}\}$; and (4) a convex conical closed function $\Phi(\mathbf{r},\mathbf{p}) = \sup\{\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) + \mathbf{r} \cdot \mathbf{K} | (\mathbf{a},\mathbf{K}) \in \mathbf{B}\}$ defined for $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$. The function Π is interpreted as an intertemporal variable profit function, while Φ is interpreted as an *ex ante* envelope profit function. We term a regular profit structure *convex* if the set $\mathbf{B}$ is convex.

A regular profit structure is termed *design linear* if the intertemporal variable profit function can be written

$$\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) = \sum_{l=1}^{L} a_l Q^l(\mathbf{p}), \tag{40}$$

where for each $l = 1,\ldots,L$, Q^l is a convex [resp., linear] conical closed function of $\mathbf{p}$ when a_l is non-negative [resp., bivalent] for all $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$.

Define a *nested profit form* by (1) a convex cone $\mathbf{F}$ of vectors $(\mathbf{r},\mathbf{p}) \in \mathbf{E}^{J+N\cdot S}$ with a non-empty interior and with r_j non-negative for $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$, $j = 1,\ldots,J$; (2) a convex cone $\mathbf{H}$ of vectors $(\mathbf{r},\mathbf{q}) \in \mathbf{E}^{J+L}$; (3) a convex conical closed function $\psi(\mathbf{r},\mathbf{q})$ on $\mathbf{H}$ which is non-increasing in r_j for all $(\mathbf{r},\mathbf{q}) \in \mathbf{H}$ and which is non-decreasing [resp., non-monotone] in q_l for l in a set of indices $\mathbf{L}_+$ [resp., $\mathbf{L}_0$]; and (4) a vector of functions $\mathbf{Q}(\mathbf{p}) = (Q^1(\mathbf{p}),\ldots,Q^L(\mathbf{p}))$ on $\mathbf{F}^p$ such that $(\mathbf{r},\mathbf{q}) \in \mathbf{H}$ if and only if there exists $\mathbf{p} \in \mathbf{F}^p$ with $\mathbf{q} = \mathbf{Q}(\mathbf{p})$ and $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$, and such that $Q^l(\mathbf{p})$ is a convex [resp., linear] conical closed function for $l \in \mathbf{L}_+$ [resp., $\mathbf{L}_0$]. The following result links design linear profit structures and nested profit forms.

Lemma 1. Consider a regular, design linear profit structure satisfying (40), and define

$$\mathbf{H} = \{(\mathbf{r},\mathbf{q}) \in \mathbf{E}^{J+L} | \mathbf{q} = \mathbf{Q}(\mathbf{p}) \text{ for some } \mathbf{p} \in \mathbf{F}^p \text{ with } (\mathbf{r},\mathbf{p}) \in \mathbf{F}\}, \tag{41}$$

$$\psi(\mathbf{r},\mathbf{q}) = \sup\{\mathbf{r}\cdot\mathbf{K} + \mathbf{q}\cdot\mathbf{a} | (\mathbf{a},\mathbf{K}) \in \mathbf{B}\} \text{ for } (\mathbf{r},\mathbf{q}) \in \mathbf{H}. \tag{42}$$

Then, $\mathbf{F}$, $\mathbf{H}$, ψ, $\mathbf{Q}$ define a nested profit form, and for $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$,

$$\Phi(\mathbf{r},\mathbf{p}) = \psi(\mathbf{r},\mathbf{Q}(\mathbf{p})). \tag{43}$$

Conversely, given a nested profit form $\mathbf{F}$, $\mathbf{H}$, ψ, $\mathbf{Q}$, define

$$\mathbf{B} = \{(\mathbf{a},\mathbf{K}) \in \mathbf{E}^{L+J} | \mathbf{q}\cdot\mathbf{a} + \mathbf{r}\cdot\mathbf{K} \leqq \psi(\mathbf{r},\mathbf{q}) \text{ for all } (\mathbf{r},\mathbf{q}) \in \mathbf{H}\}, \tag{44}$$

$$\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) = \sum_{l=1}^{L} a_l Q^l(\mathbf{p}) \quad \text{for} \quad (\mathbf{a},\mathbf{K}) \in \mathbf{B}, \tag{45}$$

and Φ satisfying (43). Then, $\mathbf{B}$, Π, Φ, $\mathbf{F}$ define a regular, convex, design linear profit structure satisfying (41) and (42).

Proof: $\mathbf{F}$ non-empty implies $\mathbf{H}$ non-empty. From the definition of $\mathbf{H}$, the right-hand side of (42) is bounded above by $\Phi(\mathbf{r},\mathbf{p})$ for some $\mathbf{p}$ with $\mathbf{q} = \mathbf{Q}(\mathbf{p})$. Hence, ψ exists on $\mathbf{H}$, and is convex conical closed by McFadden, Appendix A.3, Lemma 12.3. If $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$, then $(\mathbf{r},\mathbf{q}) \in \mathbf{H}$ for $\mathbf{q} = \mathbf{Q}(\mathbf{p})$, and $\Phi(\mathbf{r},\mathbf{p}) = \sup\{\mathbf{r}\cdot\mathbf{K} + \Pi(\mathbf{a},\mathbf{K},\mathbf{p}) | (\mathbf{a},\mathbf{K}) \in \mathbf{B}\} = \sup\{\mathbf{r}\cdot\mathbf{K} + \mathbf{a}\cdot\mathbf{Q}(\mathbf{p}) | (\mathbf{a},\mathbf{K}) \in \mathbf{B}\} = \psi(\mathbf{r},\mathbf{Q}(\mathbf{p}))$. The function $\mathbf{r}\cdot\mathbf{K} + \mathbf{a}\cdot\mathbf{Q}(\mathbf{p})$ is convex conical closed for each $(\mathbf{a},\mathbf{k}) \in \mathbf{B}$. The supremum of an arbitrary family of convex conical closed functions on $\mathbf{F}$ is again a convex conical closed function.

Next, suppose a nested profit form **F**, **H**, ψ, **Q** is given. The fundamental duality theorem for restricted profit functions (Chapter I.1, Lemma 11, Lemma 23, Theorem 24) establishes that the set **B** given by (44) is non-empty, closed, and convex and satisfies (42). The function Π defined by (45) is a sum of convex conical closed functions $a_l\mathbf{Q}^l(\mathbf{p})$, and hence also has these properties. By (42), $\psi(\mathbf{r},\mathbf{Q}(\mathbf{p})) = \sup\{\mathbf{r}\cdot\mathbf{K} + \mathbf{a}\cdot\mathbf{Q}(\mathbf{p})|(\mathbf{a},\mathbf{K}) \in \mathbf{B}\} = \sup\{\mathbf{r}\cdot\mathbf{K} + \Pi(\mathbf{a},\mathbf{K},\mathbf{p})|(\mathbf{a},\mathbf{K}) \in \mathbf{B}\}$. Hence, by the same argument as in the previous paragraph, $\psi(\mathbf{r},\mathbf{Q}(\mathbf{p}))$ is convex conical closed, and Φ defined by (43), Π, **F**, **B** define a regular convex design linear profit structure. Q.E.D.

The next result links profit structures and technological structures.

Lemma 2. Consider a regular (convex) technological structure and define

$$\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{x}|\mathbf{x} \in \mathbf{V}(\mathbf{a},\mathbf{K})\} \quad \text{for} \quad \mathbf{p} \in \mathbf{F}^p, \quad (\mathbf{a},\mathbf{K}) \in \mathbf{B}, \tag{1}$$

$$\Phi(\mathbf{r},\mathbf{p}) = \sup\{\mathbf{r}\cdot\mathbf{K} + \mathbf{p}\cdot\mathbf{x}|(\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea} \quad \text{for} \quad (\mathbf{r},\mathbf{p}) \in \mathbf{F}. \tag{3}$$

Then, **F**, **B**, Π, Φ define a regular (convex) profit structure. Conversely, given a regular (convex) profit structure, define

$$\mathbf{V}(\mathbf{a},\mathbf{K}) = \{\mathbf{x} \in \mathbf{E}^{N\cdot S}|\mathbf{p}\cdot\mathbf{x} \leqq \Pi(\mathbf{a},\mathbf{K},\mathbf{p}) \quad \text{for all} \quad \mathbf{p} \in \mathbf{F}^p\} \quad \text{for} \quad (\mathbf{a},\mathbf{K}) \in \mathbf{B}, \tag{46}$$

$$\mathbf{T}^{ea} = \{(\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{E}^{L+J+N\cdot S}|\mathbf{x} \in \mathbf{V}(\mathbf{a},\mathbf{K}), (\mathbf{a},\mathbf{K}) \in \mathbf{B}\}, \tag{47}$$

$$\mathbf{W} = \{(\mathbf{K},\mathbf{x}) \in \mathbf{E}^{J+N\cdot S}|\exists \mathbf{a} \ni (\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{T}^{ea}\} \tag{48a}$$

$$= \{(\mathbf{K},\mathbf{x}) \in \mathbf{E}^{J+N\cdot S}|\mathbf{r}\cdot\mathbf{K} + \mathbf{p}\cdot\mathbf{x} \leqq \Phi(\mathbf{r},\mathbf{p}) \text{ for all } (\mathbf{r},\mathbf{p}) \in \mathbf{F}\}. \tag{48b}$$

Then, the sets $\mathbf{V}(\mathbf{a},\mathbf{K})$ and **W** are convex, and $\mathbf{T}^{ea}$, **B**, $\mathbf{V}(\mathbf{a},\mathbf{K})$, **W**, **F** define a regular (convex) technological structure such that (1) and (3) hold.

Proof: The fundamental duality theorem for restricted profit functions (Chapter I.1, Lemma 11, Lemma 23, Theorem 24) establishes the dual properties of (1) and (46), or the dual properties of $\Phi(\mathbf{r},\mathbf{p}) = \sup\{\mathbf{r}\cdot\mathbf{K} + \mathbf{p}\cdot\mathbf{x}|(\mathbf{K},\mathbf{x}) \in \mathbf{W}\}$, equivalent to (3) and (48). The properties of $\mathbf{T}^{ea}$ defined by (47) follow from this equivalence. Q.E.D.

The next result relates design linear technological and profit structures:

Lemma 3. Suppose a regular (convex) design linear technological structure satisfies (39). Then the regular (convex) profit structure given by (1) and (3) is design linear, satisfying (40) with

$$Q^l(\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{x}|\mathbf{x} \in \mathbf{V}^l\}, \tag{49}$$

for $\mathbf{p} \in \mathbf{F}^p$, $l = 1,\ldots,L$. Conversely, suppose a regular (convex) design linear profit structure satisfies (40). Then, the regular (convex) technological structure given by (46)–(48) is design linear, satisfying (39) with

$$\mathbf{V}^l = \{\mathbf{x} \in \mathbf{E}^{N\cdot S}|\mathbf{p}\cdot\mathbf{x} \leqq Q^l(\mathbf{p}) \text{ for } \mathbf{p} \in \mathbf{F}^p\}, \tag{50}$$

for $l = 1,\ldots,L$. The $\mathbf{V}^l$ are convex, and satisfy (1), (3), (39), and (49).

Proof: Let $\mathbf{L}_+$ (resp., $\mathbf{L}_0$) denote the set of indices $l = 1,\ldots,L$, such that a_l is non-negative (resp., bivalent) for all $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$.

For a design linear technological structure,

$$\begin{aligned}\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) &= \sup\left\{\mathbf{p}\cdot\sum_{l=1}^{L} a_l\mathbf{x}^l|\mathbf{x}^l \in \mathbf{V}^l\right\} \\ &= \sum_{l\in\mathbf{L}_+} a_l \sup\{\mathbf{p}\cdot\mathbf{x}^l|\mathbf{x}^l \in \mathbf{V}^l\} \\ &\quad + \sum_{l\in\mathbf{L}_0} a_l\{\mathbf{p}\cdot\mathbf{x}^l|\mathbf{V}^l = \{\mathbf{x}^l\}\} = \sum_{l=1}^{L} a_l Q^l(\mathbf{p}),\end{aligned}$$

and (49) holds. The fundamental duality theorem applied to each $\mathbf{V}^l$ and Q^l pair and Lemma 2 establish the first conclusion.

For a design linear profit structure, the fundamental duality theorem applied to each Q^l and $\mathbf{V}^l$ pair establishes (49), $\mathbf{V}^l$ convex, and

$$\sum_{l=1}^{L} a_l Q^l(\mathbf{p}) = \sup\left\{\sum_{l=1}^{L} a_l \mathbf{p}\cdot\mathbf{x}^l|\mathbf{x}^l \in \mathbf{V}^l\right\} = \sup\left\{\mathbf{p}\cdot\mathbf{x}|\mathbf{x} \in \sum_{l=1}^{L} a_l\mathbf{V}^l\right\}.$$

The duality theorem applied to this function then establishes

$$\mathbf{V}(\mathbf{a},\mathbf{K}) = \sum_{l=1}^{L} a_l\mathbf{V}^l. \qquad \text{Q.E.D.}$$

Using the three lemmas above, we can now give the basic result which establishes that starting from the choice of any nested profit form, one can derive a netput supply system that is associated with an underlying *ex ante–ex post* technology:

Theorem 4. Consider any nested profit form $\mathbf{F}$, $\mathbf{H}$, ψ, $\mathbf{Q}$. Then:

(i) There exists a regular convex design linear technological structure $\mathbf{F}$, $\mathbf{B}$, $\mathbf{T}^{ea}$, $\mathbf{W}$, $\mathbf{V}(\mathbf{a},\mathbf{K}) = \sum_{l=1}^{L} a_l \mathbf{V}^l$, where $\mathbf{B}$ satisfies (44), the $\mathbf{V}^l$ satisfy (50), $\mathbf{T}^{ea}$ satisfies (47), and $\mathbf{W}$ satisfies (48a).

(ii) The technological structure in (i) yields an intertemporal variable profit function Π satisfying (40), (45), (1), and (46), and an *ex ante* envelope profit function Φ satisfying (43), (3), and (48b). The technological structure in (i) is the only regular convex design linear technological structure satisfying all these conditions.

(iii) Recall that $\mathbf{F}^p = \{\mathbf{p} \in \mathbf{E}^{NS} | (\mathbf{r},\mathbf{p}) \in \mathbf{F}\}$, define $\mathbf{F}^r(\mathbf{p}) = \{\mathbf{r} \in \mathbf{E}^J | (\mathbf{r},\mathbf{p}) \in \mathbf{F}\}$, and let $\mathbf{F}_0^p$ and $\mathbf{F}_0^r(\mathbf{p})$ denote the respective interiors of these sets. The partial profit functions Q^l are differentiable almost everywhere in $\mathbf{F}_0^p$. Hence, $\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) = \sum a_l Q^l(\mathbf{p})$ is differentiable almost everywhere in $\mathbf{F}_0^p$. When $\partial \Pi / \partial \mathbf{p} = \Pi_{\mathbf{p}}(\mathbf{a},\mathbf{K},\mathbf{p}) = \sum a_l Q_{\mathbf{p}}^l(\mathbf{p})$ exists, it equals the (unique) optimal *ex post* netput supply vector $\tilde{\mathbf{x}}(\mathbf{a},\mathbf{p}) = \sum_{l=1}^{L} a_l \hat{\mathbf{x}}^l(\mathbf{p})$ for this *ex ante* design and forward price vector. More generally, the subdifferential of Π with respect to $\mathbf{p}$ (see Chapter I.1) exists for all $\mathbf{p} \in \mathbf{F}_0^p$ and $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$, and each extreme vector in this subdifferential is an optimal netput supply vector in *any* regular design linear technological structure (not necessarily convex) satisfying (1), (3), (41), (42), (43) for this nested profit form.

(iv) For each $\mathbf{p} \in \mathbf{F}^p$, $\Phi(\mathbf{r},\mathbf{p}) = \psi(\mathbf{r},\mathbf{Q}(\mathbf{p}))$ is differentiable in $\mathbf{r}$ for almost all $\mathbf{r} \in \mathbf{F}_0^r(\mathbf{p})$, and $\Phi_{\mathbf{r}}(\mathbf{r},\mathbf{p}) = \psi_{\mathbf{r}}(\mathbf{r},\mathbf{Q}(\mathbf{p})) = \hat{\mathbf{K}}(\mathbf{r},\mathbf{p})$, where $\hat{\mathbf{K}}(\mathbf{r},\mathbf{p})$ is the (unique) optimal capital equipment netput vector at $(\mathbf{r},\mathbf{p})$. [A generalization analogous to that in (iii) holds for the subdifferential of Φ with respect to $\mathbf{r}$.]

(v) For almost all $(\mathbf{r},\mathbf{p}) \in \mathbf{F}_0$, Φ and the Q^l, $l = 1,\ldots,L$, are differentiable in $\mathbf{p}$, and $\hat{\mathbf{x}}(\mathbf{r},\mathbf{p}) = \Phi_{\mathbf{p}}(\mathbf{r},\mathbf{p}) = \hat{\mathbf{a}}(\mathbf{r},\mathbf{p}) \cdot \mathbf{Q}_{\mathbf{p}}(\mathbf{p})$ for any $\hat{\mathbf{a}}(\mathbf{r},\mathbf{p})$ in the subgradient of ψ with respect to $\mathbf{q}$ evaluated at $(\mathbf{r},\mathbf{Q}(\mathbf{p}))$, where $\hat{\mathbf{x}}(\mathbf{r},\mathbf{p})$ is the (unique) optimal netput supply vector for the forward price vector $\mathbf{p}$ and an optimal *ex ante* design for $(\mathbf{r},\mathbf{p})$. Each extreme vector $\hat{\mathbf{a}}(\mathbf{r},\mathbf{p})$ in the subgradient of ψ with respect to $\mathbf{q}$ is an optimal design in *any* regular design linear technological structure (not necessarily convex) satisfying (1), (3), (41), (42), (43) for this nested profit form. If ψ is differentiable with respect to $\mathbf{q}$ [and this is true for almost all $(\mathbf{r},\mathbf{q})$ in the interior of $\mathbf{H}$] at $(\mathbf{r},\mathbf{Q}(\mathbf{p}))$, then $\hat{\mathbf{a}}(\mathbf{r},\mathbf{q}) = \psi_{\mathbf{q}}(\mathbf{r},\mathbf{Q}(\mathbf{p}))$ is the (unique) optimal design.

Proof: Conclusions (i) and (ii) follow from Lemmas 1–3. The derivative properties (iii)–(v) are corollaries of Chapter I.1, Lemmas 17–19. Q.E.D.

It is convenient to summarize the formulae implied by Theorem 4 in the case that all the derivatives taken exist:

$$Q^l(\mathbf{p}) = \mathbf{p}\cdot\hat{\mathbf{x}}^l(\mathbf{p}), \tag{51}$$

$$\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) = \sum_{l=1}^{L} a_l Q^l(\mathbf{p}) = \mathbf{p}\cdot\tilde{\mathbf{x}}(\mathbf{a},\mathbf{p}) = \sum_{l=1}^{L} a_l \mathbf{p}\cdot\hat{\mathbf{x}}^l(\mathbf{p}), \tag{52}$$

$$\tilde{\mathbf{x}}_{ns}(\mathbf{a},\mathbf{p}) = \Pi_{p_{ns}}(\mathbf{a},\mathbf{K},\mathbf{p}) = \sum_{l=1}^{L} a_l Q^l_{p_{ns}}(\mathbf{p}), \tag{53}$$

$$\begin{aligned}\Phi(\mathbf{r},\mathbf{p}) = \psi(\mathbf{r},\mathbf{Q}(\mathbf{p})) &= \mathbf{r}\cdot\hat{\mathbf{K}}(\mathbf{r},\mathbf{p}) + \mathbf{p}\cdot\hat{\mathbf{x}}(\mathbf{r},\mathbf{p}) \\ &= \mathbf{r}\cdot\hat{\mathbf{K}}(\mathbf{r},\mathbf{p}) + \sum_{l=1}^{L} \hat{a}_l(\mathbf{r},\mathbf{p})\mathbf{p}\cdot\hat{\mathbf{x}}^l(\mathbf{p}),\end{aligned} \tag{54}$$

$$\hat{a}_l(\mathbf{r},\mathbf{p}) = \psi_{q_l}(\mathbf{r},\mathbf{Q}(\mathbf{p})), \tag{55}$$

$$\hat{x}_{ns}(\mathbf{r},\mathbf{p}) = \Phi_{p_{ns}}(\mathbf{r},\mathbf{p}) = \sum_{l=1}^{L} \hat{a}_l(\mathbf{r},\mathbf{p}) Q^l_{p_{ns}}(\mathbf{p}). \tag{56}$$

In econometric applications, choice of a convenient nested profit form provides a basis for empirical analysis, with formulae (51)–(56) yielding the netput supply equations (as functions of observed prices and underlying production parameters) and Theorem 4 ensuring consistency with some regular convex design linear technological structure.

5.2. *The General Case*

The results stated above for the finite case continue to hold more generally when the number of future states is infinite (as in a continuous time intertemporal model or an uncertainty model with a continuum of states of nature) or when the vector describing an *ex ante* design is infinite (as in the activity analysis model of Example 2 with an infinite set of activities). For simplicity, we confine our attention to the generalization in which the number of future states may be infinite, retaining the earlier assumption that an *ex ante* design can be described by a finite vector. This case is of some practical importance in the generation of functional forms for econometric purposes, as it allows use of the methods of calculus and differential equations.

Consider a measure space $(\mathbf{S},\mathcal{S},\mu)$ of future states, where $\mathbf{S}$ is the set of states, $\mathcal{S}$ is a σ-field of subsets of $\mathbf{S}$, and μ is a measure (a non-negative countably additive set function) defined on $\mathcal{S}$. (In applications, $\mathbf{S}$ might be an interval in the real line or a rectangle in finite-dimensional space with Lebesque measure, or a set of integers with counting measure. In the uncertainty model, μ may be the firm's subjective probability of occurrence of states of nature.)

An *ex post* production plan $\mathbf{x}$ or forward price vector $\mathbf{p}$ is a function from $\mathbf{S}$ into $\mathbf{E}^N$, hereafter assumed to be μ-measurable, i.e., $\{s \in \mathbf{S}|\mathbf{x}_s \in \mathbf{R}\} \in \mathcal{S}$ for each Borel set $\mathbf{R} \subseteq \mathbf{E}^N$. Consider a pair $(\mathbf{X},\mathbf{P})$ of linear spaces of μ-measurable functions from $\mathbf{S}$ into $\mathbf{E}^N$ such that the bilinear functional $\mathbf{p}\cdot\mathbf{x} = \int_S \mathbf{p}_s\cdot\mathbf{x}_s \, d\mu(s)$ is defined for $\mathbf{p}\in\mathbf{P},\mathbf{x}\in\mathbf{X}$. Assume the pair $(\mathbf{X},\mathbf{P})$ is separated; i.e., $\mathbf{p}\cdot\mathbf{x} = 0$ for all $\mathbf{p}\in\mathbf{P}$ implies $\mathbf{x} = \mathbf{0}$ and $\mathbf{p}\cdot\mathbf{x} = 0$ for all $\mathbf{x}\in\mathbf{X}$ implies $\mathbf{p} = \mathbf{0}$. We identify $\mathbf{X}$ as the space of *ex post* production plans, $\mathbf{P}$ as the space of forward price vectors. In applications, $\mathbf{P}$ is usually taken to be a topological space and $\mathbf{X}$ to be its adjoint. Examples of particular interest are (1) the finite case with $\mathbf{X} = \mathbf{P} = \mathbf{E}^{N\cdot S}$; (2) a case often used in problems involving uncertainty with $\mathbf{X} = \mathbf{P} = L_2(\mathbf{S},\mathcal{S},\mu,\mathbf{E}^N)$, the Hilbert space of functions from $\mathbf{S}$ into $\mathbf{E}^N$; and (3) a case occurring in intertemporal economics with the Banach spaces $\mathbf{P} = L_1(\mathbf{S},\mathcal{S},\mu,\mathbf{E}^N)$ and $\mathbf{X} = L_\infty(\mathbf{S},\mathcal{S},\mu,\mathbf{E}^N)$. We can assume, without seriously restricting potential applications, that $\mathbf{P}$ is a normed linear space and $\mathbf{X}$ is the Banach space of continuous linear functionals on $\mathbf{X}$. We shall use the weak* topology ($\mathbf{P}$-topology) on $\mathbf{X}$, denoted by $w(\mathbf{X},\mathbf{P})$, which is the weakest topology on $\mathbf{X}$ in which every functional in $\mathbf{P}$ is continuous. A generalized sequence $\mathbf{x}^d,\mathbf{d}\in\mathbf{D}$, converges to $\mathbf{x}$ in the weak* topology of $\mathbf{X}$ if and only if $\mathbf{p}\cdot\mathbf{x}^d$ converges to $\mathbf{p}\cdot\mathbf{x}$ for each $\mathbf{p}\in\mathbf{P}$. We shall use a mathematical result [Kelley and Namioka (1963, 18.6)] stating that $\mathbf{T}\subseteq\mathbf{X}$ weak* closed and bounded in norm (i.e., if $\|\mathbf{p}\|$ is the norm on $\mathbf{P}$, then $\sup_{\|\mathbf{p}\|=1} \sup_{x\in T} \mathbf{p}\cdot\mathbf{x} < +\infty$) implies $\mathbf{T}$ weak* compact.

Recall that an *ex ante* plant design is described by a vector $\mathbf{b} = (\mathbf{a},\mathbf{K})$, where $\mathbf{a}\in\mathbf{E}^L$ is an abstract design vector and $\mathbf{K}\in\mathbf{E}^J$ is a capital equipment vector with a corresponding price vector $\mathbf{r}\in\mathbf{E}^J$. Analogously to the finite case, a *technological structure* is defined by an *ex ante* envelope technology $\mathbf{T}^{ea}$ which is a non-empty set of vectors $(\mathbf{a},\mathbf{K},\mathbf{x})$ in $\mathbf{E}^{L+J} \times \mathbf{X}$, and the associated sets $\mathbf{B} \subseteq \mathbf{E}^{L+J}$, $\mathbf{V}(\mathbf{a},\mathbf{K}) \subseteq \mathbf{X}$ for $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$, $\mathbf{W} \subseteq \mathbf{E}^J \times \mathbf{X}$, and the normal cone of $\mathbf{W}$, defined as the convex set $\mathbf{F}$ of vectors $(\mathbf{r},\mathbf{p}) \in \mathbf{E}^J \times \mathbf{P}$ such that $\mathbf{r}\cdot\mathbf{K} + \mathbf{p}\cdot\mathbf{x}$ is bounded above on $\mathbf{W}$.

A technological structure is *strongly regular* if the following conditions hold: (1) the set $\mathbf{B}$ is closed, the sets $\mathbf{W}$ and $\mathbf{V}(\mathbf{a},\mathbf{K})$ for $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$ are weak* closed, and $\mathbf{V}(\mathbf{a},\mathbf{K})$ is bounded in norm; (2) capital netputs are non-positive and exhibit free disposal; and (3) for each $\mathbf{p} \in \mathbf{P}$, there exists $\delta > 0$ and $\mathbf{r} \in \mathbf{E}^J$ such that $\mathbf{r}$ is in the interior of $\mathbf{F}^r(\mathbf{p}')$ for $\mathbf{p}' \in \mathbf{P}, \|\mathbf{p}' - \mathbf{p}\| \leqq \delta$. Several comments on this definition are in order. The requirement that the *ex post* variable technology be bounded is a new condition not imposed in the finite case. Note that it is not consistent with free disposal in the *ex post* variable technology. However, it can normally be made to hold in applications by truncating the technology, carrying out the analysis below, and then reintroducing the omitted disposal activities. This condition is not essential for many of the following results; however, it greatly simplifies the mathematical arguments. Condition (3) is equivalent to the requirement that $\mathbf{F}$ have a non-empty interior in the norm topology of $\mathbf{E}^J \times \mathbf{P}$. A sufficient condition for (3) to hold is that the "average product" of capital go to zero when capital inputs are unbounded; i.e., $\lim_{|\mathbf{K}|\to\infty} \sup_{x,a}\{\|\mathbf{x}\|_x/|\mathbf{K}| \mid \mathbf{x} \in \mathbf{V}(\mathbf{a},\mathbf{K}),(\mathbf{a},\mathbf{K}) \in \mathbf{B}\} = 0$, where $\|\mathbf{x}\|_x$ is the norm of the Banach space $\mathbf{X}$ and $|\mathbf{K}|$ is the Euclidean norm on $\mathbf{E}^J$.

A technological structure is *convex* if $\mathbf{T}^{ea}$ is convex. It is *design linear* when the *ex post* variable technology has the form

$$\mathbf{V}(\mathbf{a},\mathbf{K}) = \sum_{l=1}^{L} a_l \mathbf{V}^l,$$

where the $\mathbf{V}^l$ are non-empty, weak* closed, and bounded in norm; and for each $l = 1,\ldots,L$, either a_l is non-negative for all $(\mathbf{a},\mathbf{K}) \in \mathbf{B}$ or $\mathbf{V}^l$ is a singleton.

Define a *strong nested profit form* by (1) a convex cone $\mathbf{F}$ of vectors $(\mathbf{r},\mathbf{p}) \in \mathbf{E}^J \times \mathbf{P}$ with r_j non-negative for $(\mathbf{r},\mathbf{p}) \in \mathbf{F}, j = 1,\ldots,J$, and such that for each $\mathbf{p} \in \mathbf{P}$ there exists $\delta > 0$ and $\mathbf{r} \in \mathbf{E}^J$ such that $\mathbf{r}$ is in the interior of $\mathbf{F}^r(\mathbf{p}')$ for $\mathbf{p}' \in \mathbf{P}, \|\mathbf{p} - \mathbf{p}'\| \leqq \delta$; (2) a convex cone $\mathbf{H}$ of vectors $(\mathbf{r},\mathbf{q}) \in \mathbf{E}^{J+L}$; (3) a convex conical closed function $\psi(\mathbf{r},\mathbf{q})$ on $\mathbf{H}$ which is non-increasing in r_j for all $(\mathbf{r},\mathbf{q}) \in \mathbf{H}$ with $r_j \geqq 0$ and which is non-decreasing [resp., non-monotone] in q_l for l in a set of indices $\mathbf{L}_+$ [resp., $\mathbf{L}_0$]; and (4) a vector of functions $\mathbf{Q}(\mathbf{p}) = (Q^1(\mathbf{p}),\ldots,Q^L(\mathbf{p}))$ on $\mathbf{P}$ such that $(\mathbf{r},\mathbf{q}) \in \mathbf{H}$ if and only if there exists $\mathbf{p} \in \mathbf{P}$ with $\mathbf{q} = \mathbf{Q}(\mathbf{p})$ and $(\mathbf{r},\mathbf{p}) \in \mathbf{F}$, and such that $Q^l(\mathbf{p})$ is a convex [resp., linear] conical uniformly Lipschitz function for $l \in \mathbf{L}_+$ [resp., $\mathbf{L}_0$].

The following result extends the conclusions of Theorem 4 to the general case where $\mathbf{S}$ need not be finite:

Theorem 5. Suppose one is given a strong nested profit form $\mathbf{F}$, $\mathbf{H}$, ψ, $\mathbf{Q}$, $\mathbf{L}_+$, $\mathbf{L}_0$. Define

$$\mathbf{B} = \{(\mathbf{a},\mathbf{K}) \in \mathbf{E}^{L+J} | \mathbf{r}\cdot\mathbf{K} + \mathbf{q}\cdot\mathbf{a} \leqq \psi(\mathbf{r},\mathbf{q}) \text{ for all } (\mathbf{r},\mathbf{q}) \in \mathbf{H}\}, \tag{57}$$

$$\mathbf{V}^l = \{\mathbf{x} \in \mathbf{X} | \mathbf{p}\cdot\mathbf{x} \leqq Q^l(\mathbf{p}) \text{ for all } \mathbf{p} \in \mathbf{P}\}, \tag{58}$$

$$\mathbf{T}^{ea} = \left\{(\mathbf{a},\mathbf{K},\mathbf{x}) \in \mathbf{E}^{L+J} \times \mathbf{X} | \mathbf{x} \in \sum_{l=1}^{L} a_l \mathbf{V}^l \text{ and } (\mathbf{a},\mathbf{K}) \in \mathbf{B}\right\}. \tag{59}$$

Then (57)–(59) define a strongly regular convex design linear technological structure. This structure satisfies

$$Q^l(\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{x} | \mathbf{x} \in \mathbf{V}^l\}, \tag{60}$$

$$\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) = \sum_{l=1}^{L} a_l Q^l(\mathbf{p}) \quad \text{for} \quad \mathbf{p} \in \mathbf{P}, \tag{61}$$

$$\psi(\mathbf{r},\mathbf{q}) = \sup\{\mathbf{r}\cdot\mathbf{K} + \mathbf{q}\cdot\mathbf{a} | (\mathbf{a},\mathbf{K}) \in \mathbf{B}\} \quad \text{for} \quad (\mathbf{r},\mathbf{q}) \in \mathbf{H}, \tag{62}$$

$$\Phi(\mathbf{r},\mathbf{p}) = \psi(\mathbf{r},\mathbf{Q}(\mathbf{p})) \quad \text{for} \quad (\mathbf{r},\mathbf{p}) \in \mathbf{F}, \tag{63}$$

and is the only strongly regular convex design linear technological structure satisfying (60)–(63).

Proof: Except for the duality conditions (58) and (60), the statements of this theorem follow from the arguments of Lemmas 1–3. We first show that the $\mathbf{V}^l$ defined by (58) are non-empty, convex, bounded in norm, and weak* closed. We consider $l \in \mathbf{L}_+$; the remaining case is left to the reader. Define $\mathbf{epi}\, Q^l = \{(\mathbf{p},\mathbf{q}) \in \mathbf{P} \times \mathbf{E} | q \geqq Q^l(\mathbf{p})\}$ and $\mathbf{G}^l = \{(\mathbf{x},\xi) \in \mathbf{X} \times \mathbf{E} | -\mathbf{p}\cdot\mathbf{x} + \xi q \geqq 0 \text{ for } (\mathbf{p},\mathbf{q}) \in \mathbf{epi}\, Q^l\}$. Since Q^l is Lipschitz and conical, $\mathbf{epi}\, Q^l$ is a closed (in norm) convex cone with $(0,-1) \notin \mathbf{epi}\, Q^l$, $\mathbf{G}^l$ and $\mathbf{epi}\, Q^l$ are polar cones, and $\mathbf{G}^l$ is a weak* closed convex cone. $\mathbf{G}^l$ contains non-zero vectors [Dunford and Schwartz (1958, V.9.8)], and $\mathbf{0} \neq (\mathbf{x},\xi) \in \mathbf{G}^l$ implies $\xi > 0$. Then, $\mathbf{V}^l = \{\mathbf{x} \in \mathbf{X} | (\mathbf{x},1) \in \mathbf{G}^l\}$ satisfies (58) and is non-empty, convex, and weak* closed. Since Q^l satisfies $|Q^l(\mathbf{p})| \leqq m\|\mathbf{p}\|$ for some $m > 0, \mathbf{x} \in \mathbf{V}^l$ satisfies $|\mathbf{p}\cdot\mathbf{x}| \leqq m\|\mathbf{p}\|$ for all $\mathbf{p} \in \mathbf{P}$, or $\|\mathbf{x}\| \leqq m$. Hence, $\mathbf{V}^l$ is bounded in norm. Equation (60) and the uniqueness statement of the theorem follow from the polarity of the cones **epi** Q^l and $\mathbf{G}^l$. Q.E.D.

As in the finite case, one may define the subdifferential of Q^l at $\mathbf{p}$ to be the set of points $\mathbf{x} \in \mathbf{X}$ satisfying $\mathbf{p}'\cdot\mathbf{x} \leqq Q^l(\mathbf{p}+\mathbf{p}') - Q^l(\mathbf{p})$ for $\mathbf{p}' \in \mathbf{P}$. The assumption that Q^l is convex conical and uniformly Lipschitz implies the subdifferential is always non-empty and equals the set of $\mathbf{x} \in \mathbf{V}^l$ maximizing $\mathbf{p}\cdot\mathbf{x}'$ on $\mathbf{V}^l$. The almost everywhere differentiability of Q^l in the finite case does not carry over to the general infinite-dimensional case; however, the following partial generalization holds. First, several definitions: A subspace $\mathbf{P}_0$ of $\mathbf{P}$ is *separable* if it contains a countable dense subset. Sufficient conditions for $\mathbf{P}_0$ to be separable are (i) that it be the closed space spanned by a countable subset of $\mathbf{P}$ [Dunford and Schwartz (1958, II.1.5)] or (ii) that $\mathbf{S}$ be a compact metric space and $\mathbf{P}_0$ be a closed space of continuous functions on $\mathbf{S}$ [Dunford and Schwartz (1958, V.7.12)]. $\mathbf{P}$ is the *direct sum* of subspaces $\mathbf{P}_0$ and $\mathbf{P}_1$, written $\mathbf{P} = \mathbf{P}_0 \oplus \mathbf{P}_1$, if each $\mathbf{p} \in \mathbf{P}$ has a unique representation $\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1, \mathbf{p}_0 \in \mathbf{P}_0, \mathbf{p}_1 \in \mathbf{P}_1$. Define $\mathbf{X}_0 = \{\mathbf{x} \in \mathbf{X} | \mathbf{p}\cdot\mathbf{x} = 0 \text{ for } \mathbf{p} \in \mathbf{P}_0\}$ and the quotient space $\mathbf{X}/\mathbf{X}_0$. A function $\mathbf{Q}:\mathbf{P}_0 \to \mathbf{E}^L$ is *differentiable* at $\mathbf{p}_0 \in \mathbf{P}_0$ if there exists a unique $\mathbf{x}_0^l \in \mathbf{X}/\mathbf{X}_0$ such that $\mathbf{p}'\cdot\mathbf{x}_0^l = \lim_{\theta\to 0^+}[\tilde{Q}^l(\mathbf{p}_0+\mathbf{p}') - \tilde{Q}^l(\mathbf{p}_0)]/\theta$ for all $\mathbf{p}' \in \mathbf{P}_0$, $l = 1,\ldots,L$.

Lemma 6. Suppose $\mathbf{P} = \mathbf{P}_0 \oplus \mathbf{P}_1$, where $\mathbf{P}_0$ is a closed separable subspace of $\mathbf{P}$. For each $\mathbf{p}_1 \in \mathbf{P}_1$, the convex conical uniformly Lipschitz functions $\mathbf{Q}(\mathbf{p}_0+\mathbf{p}_1) = (Q^1(\mathbf{p}_0+\mathbf{p}_1),\ldots,Q^L(\mathbf{p}_0+\mathbf{p}_1))$, considered as functions of $\mathbf{p}_0 \in \mathbf{P}_0$ are differentiable on a dense subset of $\mathbf{P}_0$.

Proof: Define $q^l:\mathbf{P}_0 \to \mathbf{E}$ by $q^l(\mathbf{p}_0) = Q^l(\mathbf{p}_0+\mathbf{p}_1)$ for $l \in L$, and define $\mathbf{q}(\mathbf{p}_0) = \sum_{l=1}^L q^l(\mathbf{p}_0)$. Consider the tangent functional $\tau^l(\mathbf{p}_0,\mathbf{p}') = \lim_{\theta\to 0^+}(q^l(\mathbf{p}_0+\theta\mathbf{p}') - q^l(\mathbf{p}_0))/\theta, \mathbf{p}' \in \mathbf{P}_0$, and note that the tangent functional $\tau(\mathbf{p}_0,\mathbf{p}')$ of $q(\mathbf{p}_0)$ satisfies $\tau(\mathbf{p}_0,\mathbf{p}') = \sum_{l=1}^L \tau^l(\mathbf{p}_0,\mathbf{p}')$. Note that $\mathbf{q}$ is uniformly Lipschitz on $\mathbf{P}_0$, and that $\tau^l(\mathbf{p}_0,\mathbf{p}') \geqq -\tau^l(\mathbf{p}_0,-\mathbf{p}')$. The proof of V.9.8 in Dunford and Schwartz (1958) establishes that for $\mathbf{p}_0$ in a dense subset $\mathbf{P}_0'$ of $\mathbf{P}_0, \tau(\mathbf{p}_0,\mathbf{p}') = -\tau(\mathbf{p}_0,-\mathbf{p}')$ for $\mathbf{p}' \in \mathbf{P}_0$. From the inequality above, this implies $\tau^l(\mathbf{p}_0,\mathbf{p}') = -\tau^l(\mathbf{p}_0,-\mathbf{p}')$ for $\mathbf{p}_0 \in \mathbf{P}_0', \mathbf{p}' \in \mathbf{P}_0, l = 1,\ldots,L$. Then, $\tau^l(\mathbf{p}_0,\mathbf{p}') = \mathbf{p}'\cdot\mathbf{x}_0$, where $\mathbf{x}_0$ is in the subdifferential of $Q^l(\mathbf{p})$ at $\mathbf{p} = \mathbf{p}_0+\mathbf{p}_1$, implying $\mathbf{x}_0$ is unique in $\mathbf{X}/\mathbf{X}_0$. Q.E.D.

6. Separable Technology Across States

The abstract model of *ex ante–ex post* firm behavior presented in Sections 3 and 5 requires no specific assumptions on the structure of *ex post* technologies across future states. However, each of the examples in Section 4 is based on a technology structure that is "separable across states". While we wish to avoid imposing unnecessary structure that precludes possible applications, it is important to explore the implications of this conventional structural assumption of separability ("non-joint" production) across future states.[9] This condition reduces substantially the "dimension" of the intertemporal profit function, making it an important property in empirical applications, where this function must be given an explicit form.

An *ex post* variable technology $V(\mathbf{b}),\mathbf{b} \in \mathbf{B}$, is *separable across states* if for each future state $s \in \mathbf{S}$ there exists a set $\mathbf{V}_s(\mathbf{b})$ of N-vectors of netputs $\mathbf{x}_s$, termed the *ex post* s-technology, such that

$$\mathbf{V}(\mathbf{b}) = \{\mathbf{x} \in X | \mathbf{x}_s \in \mathbf{V}_s(\mathbf{b}) \text{ for } s \in \mathbf{S}\}.$$

When S is finite, $\mathbf{V}(\mathbf{b})$ is just the Cartesian product of the s-technologies. The important characteristic of this structure is that, given the *ex post* technology, the set of possible netput vectors in a state s is independent of the operating points chosen in other states. For the model with uncertainty and one period of operation, this condition will always be imposed (recall that decisions made before the state of nature is known are described in $\mathbf{b}$). An example illustrates application of this condition in the model with intertemporal variation and no uncertainty. Suppose a firm chooses *ex ante* an input level K of one producer durable, and in each future period $s = 1,...,L$ chooses a labor input L_s and output Y_s satisfying a production function $Y_s \leqq (Kd_s)^{\alpha} L_s^{\beta}$, where d_s is a depreciation effect. If d_s is exogenous, resulting from weathering depreciation, then this technology is separable across states. Alternately, if d_s depends on output levels in previous periods because of wear and tear depreciation, this separable structure does not hold.[10]

[9]A quick accounting of problems employing intertemporal production yields the following list of phenomena related to intertemporal structure: disembodied technical change, weathering depreciation of durable equipment, wear and tear depreciation of durable equipment, learning-by-doing in the plant, endogenous construction rate and scrapping decisions, expansion and modernization decisions. The first two phenomena are consistent with intertemporal separability, the remainder are not.

[10]In a form of the separability assumption that appears in the economic growth literature, inputs in one period yield outputs in the following period in two-period

For each *ex post* s-technology $\mathbf{T}_s(\mathbf{b})$ and forward price vector $\mathbf{p} = (\mathbf{p}_s : s \in \mathbf{S})$, define an *s-future profit function*,

$$\Pi_s(\mathbf{b},\mathbf{p}_s) = \sup\{\mathbf{p}_s \cdot \mathbf{x}_s | \mathbf{x}_s \in \mathbf{T}_s(\mathbf{b})\}. \tag{64}$$

The future profit function is then given by the sum $\Pi(\mathbf{b},\mathbf{p}) = \sum_{s \in S} \Pi_s(\mathbf{b},\mathbf{p}_s) = \sum_{s \in S} \delta_s \Pi_s(\mathbf{b},\tilde{\mathbf{p}}_s)$ when S is finite, and more generally by the integral $\Pi(\mathbf{b},\mathbf{p}) = \int_S \Pi_s(\mathbf{b},\mathbf{p}_s) \mathrm{d}\mu(s)$. Conversely, additive separability of the profit function across states implies that the *ex post* technology is, in effect, separable across states; the precise implication for S finite is

$$\begin{aligned}\text{convex hull } (\mathbf{V}(\mathbf{b})) &= \{\mathbf{x} | \mathbf{p}\cdot\mathbf{x} \leqq \Pi(\mathbf{b},\mathbf{p}) \text{ for all } \mathbf{p}\} \\ &= \{\mathbf{x} | \textstyle\sum_{s\in S} \mathbf{p}_s \cdot \mathbf{x}_s \leqq \sum_{s \in S} \Pi_s(\mathbf{b},\mathbf{p}_s) \text{ for all } \mathbf{p}\} \\ &= \times_{s \in S} \{\mathbf{x}_s | \mathbf{p}_s \cdot \mathbf{x}_s \leqq \Pi_s(\mathbf{b},\mathbf{p}_s) \text{ for all } \mathbf{p}_s\} \\ &= \times_{s \in S} \text{ convex hull } (V_s(\mathbf{b})).\end{aligned}$$

In applications, the s-technologies are frequently assumed to vary in a simple pattern over states. A first example in the intertemporal variation model is the "one-hoss-shay" technology, with $\mathbf{T}_s(\mathbf{b}) = \mathbf{T}_1(\mathbf{b})$ for $s = 1,\dots,L$. A second example giving a uniform s-technology across states is the single-period model in which the firm faces uncertain market prices, but a certain technology $\mathbf{T}_1(\mathbf{b})$. The profit function then satisfies $\Pi(\mathbf{b},\mathbf{p}) = \int_S \Pi_1(\mathbf{b},\mathbf{p}_s)\mathrm{d}\mu(s)$. The following transformation of this second example will be useful in applications: Let $\mathbf{p}_.$ denote an N-vector of prices and define $\tilde{S}(\mathbf{p}_.) = \{s \in S | \tilde{\mathbf{p}}_s \leqq \mathbf{p}_.\}$ and $G(\mathbf{p}_.) = \int_{\tilde{S}(\mathbf{p}_.)} \delta_s \,\mathrm{d}\mu(s)$. Then, G is the distribution function of current price vectors, and $\Pi(\mathbf{b},\mathbf{p}) = \int_{E^N} \Pi_1(\mathbf{b},\mathbf{p}_.)\mathrm{d}G(\mathbf{p}_.)$ is the expected value of current profit. Since the profit function $\Pi_1(\mathbf{b},\mathbf{p}_.)$ is convex in $\mathbf{p}_.$, we have the implication $\int_{E^N} \Pi_1(\mathbf{b},\mathbf{p}_.)\mathrm{d}G(\mathbf{p}_.) \geqq \Pi_1(\mathbf{b},\int_{E^N} \mathbf{p}_.\, \mathrm{d}G(\mathbf{p}_.))$, with equality holding if the $\mathbf{p}_s$ are proportional in all states. First, we conclude that when relative prices are not affected by the state of nature, the *ex post* operation of the plant is reduced to the problem of maximizing current profit at expected prices. Second, that given the certain *ex post* technology $\mathbf{T}_1(\mathbf{b})$, the firm cannot lose and may gain from increased uncertainty. To make

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technologies. This recursive technology structure assumes the existence of markets for all intermediate goods, including rental and second-hand markets for producer durables. This structure can be included in the separable across-states assumption in our analysis by treating each state s as made up of two subperiods, with the second subperiod of state s coinciding chronologically with the first subperiod of the successive state. The markets for inputs and outputs at the same chronological time are then treated as distinct markets (with arbitrage).

this statement more precise, we consider a comparative argument in which commodities have a current price vector $\mathbf{p}_.$ with distribution $G(\mathbf{p}_.)$ in the first case and a current price vector $\mathbf{p}_. + \mathbf{p}'_.$ in the second, and $\mathbf{p}'_.$ introduces a "pure spread" in the price distribution; i.e., $\mathbf{p}'_.$ has a conditional distribution $H(\mathbf{p}'_.|\mathbf{p}_.)$ with $\int_{\mathbf{E}^N} \mathbf{p}'_.\, \mathrm{d}H(\mathbf{p}'_.|\mathbf{p}_.) = 0$. Then, $\int_{\mathbf{E}^N}(\int_{\mathbf{E}^N} \Pi_1(\mathbf{b},\mathbf{p}_. + \mathbf{p}'_.)\mathrm{d}H(\mathbf{p}'_.|\mathbf{p}_.))\mathrm{d}G(\mathbf{p}_.) \geqq \int_{\mathbf{E}^N} \Pi_1(\mathbf{b},\mathbf{p}_. + \int_{\mathbf{E}^N} \mathbf{p}'_.\, \mathrm{d}H(\mathbf{p}'_.|\mathbf{p}_.))\mathrm{d}G(\mathbf{p}_.) = \int_{\mathbf{E}^N} \Pi_1(\mathbf{b},\mathbf{p}_.)\mathrm{d}G(\mathbf{p}_.)$, and expected profits from the "spread" prices $\mathbf{p}_. + \mathbf{p}'_.$ are at least as high as those from the prices $\mathbf{p}_.$. An intuitive justification of this conclusion can be given using Figure 13. Suppose an industry with identical firms faces an uncertain demand for industry output Y, each of the curves D^1 and D^2 occurring with probability one-half, and yielding an expected output price $\bar{p}_.$. Supply at the expected output price will be greater than (less than) the expected value of output in the case of a convex industry supply in Figure 13(a) [a concave industry supply in Figure 13(b)]. However, because higher equilibrium prices always induce higher supply, and hence higher profits, the expected value of profit will be unambiguously higher than profit at the expected output price in either case (a) or (b).

A structure of the s-technologies across states only slightly less simple than the uniform examples considered above is the case of exogenous commodity augmentation in which $V_s(\mathbf{b}) = \{(x_1A_{1s},\dots,x_NA_{Ns})|(x_1,\dots,x_N) \in V_1(\mathbf{b})\}$, the A_{ns} being exogenous non-negative numbers. In the model with intertemporal production, the A_{ns} may represent the effects of weathering depreciation and disembodied technical change. In the single-period model with uncertainty, the A_{ns} reflect the quality of commodities in various states of nature. In the concrete example with a single durable input K and production of output Y_1 in state 1 from input L_1 satisfying $Y_1 \leqq K^\alpha L_1^\beta$, this structure gives a production function $Y_s \leqq K^\alpha L_s^\beta(A_{Ls}^\beta A_{Ys}^{-1})$. The commodity-augmenting structure implies that the variable profit function can be written $\Pi(\mathbf{b},\mathbf{p}) = \int_S \Pi_1(K,p_{1s}/A_{1s},\dots,p_{Ns}/A_{Ns})\mathrm{d}\mu(s)$, with p_{ns}/A_{ns} interpreted as an efficiency forward price.

A final comment is in order on the differentiability of $\Pi(\mathbf{a},\mathbf{K},\mathbf{p})$ in the general case of a measure space $(\mathbf{S},\mathcal{S},\mu)$ of future states when the technology is separable across states. If each s-technology $V_s(\mathbf{a},\mathbf{K})$ is strictly convex in $\mathbf{E}^N$, then $\Pi_s(\mathbf{a},\mathbf{K},\mathbf{p}_s)$ is differentiable in $\mathbf{p}_s$ on $\mathbf{E}^N$. Hence, $\mathbf{p}'\cdot\hat{\mathbf{x}} = \int_S(\partial\Pi_s/\partial\mathbf{p}_s)\cdot\mathbf{p}'_s\mathrm{d}\mu(s)$ for $\mathbf{p}' \in \mathbf{P}$, $\hat{\mathbf{x}} = (\hat{\mathbf{x}}_s : s \in \mathbf{S})$, with $\hat{\mathbf{x}}_s$ the unique optimal netput vector in $\mathbf{V}_s(\mathbf{a},\mathbf{K})$ at price vector $\mathbf{p}_s$. Hence, $\Pi(\mathbf{a},\mathbf{K},\mathbf{p})$ is differentiable in $\mathbf{p}$ on $\mathbf{P}$.

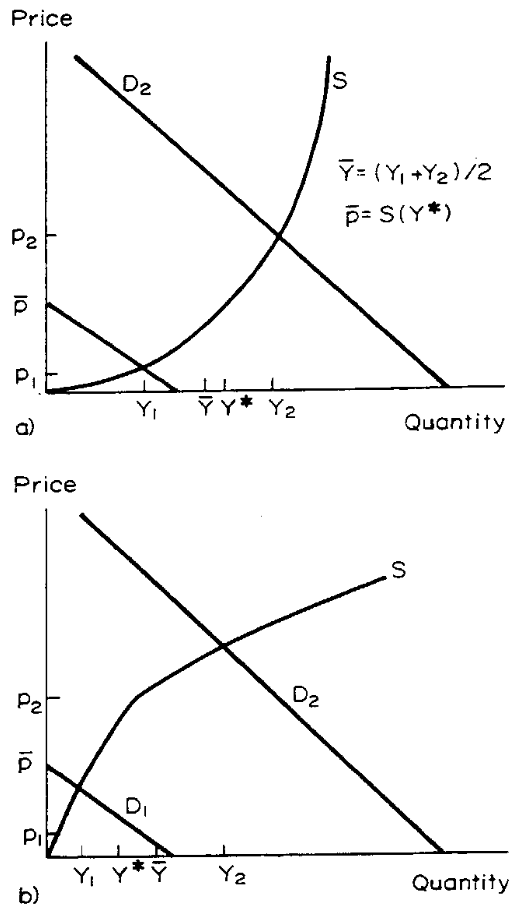

FIGURE 13

7. A General Linear-in-Parameters *Ex Ante–Ex Post* Technology

The algorithm introduced in Section 5 provides a general procedure for generating two-level technologies. Example 4 in Section 4 illustrates a construction for the case of cost minimization that is linear in the underlying production parameters, making it particularly convenient for statistical analysis. We now present a generalization of this family of nested forms to the profit-maximization case, and show that this generalization is robust in the sense that locally it can mimic the net supply behavior of a broad class of two-level technologies. For simplicity, we assume in this analysis that the set of states **S** is finite.

Consider a nested profit form as defined in Section 5, described by **F**,

$\mathbf{H}$, ψ, and $\mathbf{Q} = (Q^1,...,Q^L)$. Suppose ψ has a linear-in-parameters form

$$\psi(\mathbf{r},\mathbf{q}) = \sum_{g=1}^{G} b_g R^g(\mathbf{r},\mathbf{g}), \tag{65}$$

where b_g is non-negative and R^g is non-decreasing in $\mathbf{q}$ for $(\mathbf{r},\mathbf{q}) \in \mathbf{H}$. Then, Theorem 4 holds, implying equations (51)–(56) can be used to specify the net supply system. Rewriting (53)–(56) for the functional form (65),

$$\Phi(\mathbf{r},\mathbf{p}) = \psi(\mathbf{r},Q(\mathbf{p})) = \sum_{g=1}^{G} b_g R^g(\mathbf{r},Q(\mathbf{p})), \tag{66}$$

$$\hat{a}_l(\mathbf{r},\mathbf{p}) = \sum_{g=1}^{G} b_g R^g_{q_l}(\mathbf{r},Q(\mathbf{p})), \tag{67}$$

$$\hat{x}_{ns}(\mathbf{r},\mathbf{p}) = \sum_{l=1}^{L} \hat{a}_l(\mathbf{r},\mathbf{p}) Q^l_{p_{ns}}(\mathbf{p}) = \sum_{g=1}^{G} b_g \left[\sum_{l=1}^{L} R^g_{q_l}(\mathbf{r},Q(\mathbf{p}))Q^l_{p_{ns}}(\mathbf{p})\right], \tag{68}$$

Note that the system (66)–(68) is linear in the underlying technological parameters b_g.

Adapting the functional form of Example 4, let $\mathbf{N}$, $\mathbf{J}$, and $\mathbf{S}$ denote the sets of indices $\{1,...,N\}$, $\{1,...,J\}$, and $\{1,...,S\}$, respectively, and define

$$Q^{ijst}(\mathbf{p}) = -(p_{is}p_{jt})^{1/2}, \tag{69}$$

$$\bar{Q}^{ij}(\mathbf{p}) = -\sum_{s\in\mathbf{S}} (p_{is}p_{js})^{1/2}, \tag{70}$$

for $i,j \in \mathbf{N}$ and $s,t \in \mathbf{S}$. Suppose ψ has a linear-in-parameters form

$$\begin{aligned}\psi(\mathbf{r},\mathbf{q}) = &\sum_{i,j,k,l\in\mathbf{N}} b^1_{ij,kl}[-(q_{ij}q_{kl})^{1/2}] + 2\sum_{\substack{i,j\in\mathbf{N}\\k\in\mathbf{J}}} b^2_{ij,k}[-(-q_{ij}r_k)^{1/2}] \\ &+ \sum_{\substack{ijst\in\mathbf{L}\\kluv\in\mathbf{L}}} b^3_{ijst,kluv}[-(q_{ijst}q_{kluv})^{1/2}] \\ &+ 2\sum_{\substack{ijst\in\mathbf{L}\\k\in\mathbf{J}}} b^4_{ijst,k}[-(-q_{ijst}r_k)^{1/2}] + \sum_{k,l\in\mathbf{J}} b^5_{kl}[-(r_kr_l)^{1/2}],\end{aligned} \tag{71}$$

where $\mathbf{L} = \{(ijst)|i,j \in \mathbf{N}; s,t \in \mathbf{S}\}$ and the underlying b parameters satisfy the following conditions:

(1) $b^1_{ij,kl}$ symmetric under permutation of i and j, of (ij) and (kl), and of combinations of these permutations; and non-negative unless $(ij) = (kl)$ for some permutation of i and j;

(2) $b^2_{ij,k}$ symmetric under permutation of i and j; and non-negative;
(3) $b^3_{ijst,kluv}$ symmetric under permutation of i and j, of s and t, or $(ijst)$ and $(kluv)$, and of combinations of these permutations; and non-negative unless $(ijst) = (kluv)$ for some permutation of i and j, and of s and t;
(4) $b^4_{ijst,k}$ symmetric under permutation of i and j, or s and t, of combinations of these permutations; and non-negative;
(5) b^5_{kl} symmetric under permutation of k and l, and non-negative unless $k = l$.

The set $\mathbf{H}$ is defined so that $\mathbf{q}$ is non-positive and ψ is non-decreasing in $\mathbf{q}$ at $(\mathbf{r},\mathbf{q}) \in \mathbf{H}$; this implies a constraint on the range of the "diagonal" b parameters. One may readily verify that this system meets the requirements of a nested profit form. Writing out the net supply system:

$$\hat{a}_{ij}(\mathbf{r},\mathbf{p}) = -\sum_{k,l\in\mathbf{N}} b^1_{ij,kl}(\tilde{Q}^{kl}(\mathbf{p})/\tilde{Q}^{ij}(\mathbf{p}))^{1/2} - \sum_{k\in\mathbf{J}} b^2_{ij,k}(-r_k/\tilde{Q}^{ij}(\mathbf{p}))^{1/2}, \tag{72}$$

$$\hat{a}_{ijst}(\mathbf{r},\mathbf{p}) = -\sum_{kluv\in\mathbf{L}} b^3_{ijst,kluv}(Q^{kluv}(\mathbf{p})/Q^{ijst}(\mathbf{p}))^{1/2} - \sum_{k\in\mathbf{J}} b^4_{ijst,k}(-r_k/Q^{ijst}(\mathbf{p}))^{1/2}, \tag{73}$$

$$\hat{K}_k(\mathbf{r},\mathbf{p}) = -\sum_{l\in\mathbf{J}} b^5_{kl}(r_l/r_k)^{1/2} - \sum_{i,j\in\mathbf{N}} b^2_{ij,k}(-\tilde{Q}^{ij}(\mathbf{p})/r_k)^{1/2} - \sum_{ijst\in\mathbf{L}} b^4_{ijst,k}(-Q^{ijst}(\mathbf{p})/r_k)^{1/2}, \tag{74}$$

$$\hat{x}_{is}(\mathbf{r},\mathbf{p}) = -\sum_{j\in\mathbf{N}} \hat{a}_{ij}(p_{js}/p_{is})^{1/2} - \sum_{\substack{j\in\mathbf{N}\\ t\in\mathbf{S}}} \hat{a}_{ijst}(p_{jt}/p_{is})^{1/2}, \tag{75}$$

$$\begin{aligned}\hat{x}_{is}(\mathbf{r},\mathbf{p}) = &\sum_{j,k,l\in\mathbf{N}} b^1_{ij,kl}(p_{js}/p_{is})^{1/2}(\tilde{Q}^{kl}(\mathbf{p})/\tilde{Q}^{ij}(\mathbf{p}))^{1/2} \\ &+ \sum_{\substack{j\in\mathbf{N}\\ k\in\mathbf{J}}} b^2_{ij,k}(p_{js}/p_{is})^{1/2}(-r_k/\tilde{Q}^{ij}(\mathbf{p}))^{1/2} \\ &+ \sum_{\substack{j\in\mathbf{N}\\ t\in\mathbf{S}\\ kluv\in\mathbf{L}}} b^3_{ijst,kluv}(p_{jt}/p_{is})^{1/2}(Q^{kluv}(\mathbf{p})/Q^{ijst}(\mathbf{p}))^{1/2} \\ &+ \sum_{\substack{j\in\mathbf{N}\\ t\in\mathbf{S}\\ k\in\mathbf{J}}} b^4_{ijst,k}(p_{jt}/p_{is})^{1/2}(-r_k/Q^{ijst}(\mathbf{p}))^{1/2}.\end{aligned} \tag{76}$$

Given data on a cross-section of firms operated in each future state, the system (74), (76) can be estimated by multivariate regression methods. It is of interest to state explicitly in terms of the b parameters some of the important hypotheses imposed on the *ex ante–ex post* technology:

(1) Separable across states *ex post* variable technology. This hypothesis holds if and only if the *ex post* variable profit function is additively separable across states, or $\hat{a}_{ijst} = 0$ for $s \neq t$. This is equivalent to the linear hypothesis $b^3_{ijst,kluv} = 0$ unless $s = t$ and $u = v$ and $b^4_{ijst,k} = 0$ unless $s = t$.

(2) Separable and uniform across states *ex post* variable technology. This hypothesis holds if and only if (1) above holds and $\hat{a}_{ijss} = \hat{a}_{ij11}$. This is equivalent to the linear hypothesis $b^3_{ijst,kluv} = 0 = b^4_{ijst,k}$, since the equality above holds only if $\hat{a}_{ij11} \equiv 0$.

(3) Putty–clay hypothesis. This condition holds if and only if $\hat{\hat{a}}_{ij} = 0$ for $i \neq j$ and $\hat{a}_{ijst} = 0$ for $i \neq j$ or $s \neq t$. This is equivalent to the linear hypothesis $b^1_{ij,kl} = 0$ unless $i = j$, $k = l$; $b^2_{ij,k} = 0$ unless $i = j$; $b^3_{ijst,kluv} = 0$ unless $i = j$, $k = l$, $s = t$, $u = v$; and $b^4_{ijst,k} = 0$ unless $i = j$, $s = t$.

(4) Non-jointness of net supplies (is) and (jt) in the *ex post* variable technology. If $s \neq t$, this hypothesis is equivalent to the linear hypothesis $b^3_{ijst,kluv} = 0$ for $kluv \in \mathbf{L}$ and $b^4_{ijst,k} = 0$ for $k \in \mathbf{J}$. If $s = t$, this hypothesis is equivalent to $b^1_{ij,kl} = 0$ for $k,l \in \mathbf{N}$, $b^2_{ij,k} = 0$ for $k \in \mathbf{J}$, $b^3_{ijss,kluv} = 0$ for $kluv \in \mathbf{L}$, and $b^4_{ijss,k} = 0$ for $k \in \mathbf{J}$.

Finally, we note that the nested profit form (69)–(71) has the following approximation property, established in Chapter II.2: Consider any nested profit form which is twice continuously differentiable in the neighborhood of a point and which has at this point the *gross substitutes* property that the mixed second partial derivatives of ψ and the Q^m are non-positive. Then there exist b parameters in the form (69)–(71) such that this form agrees with the nested profit function through second-order partials at this point. Hence, the system (69)–(71) can mimic locally the net supplies and elasticities of any *ex ante–ex post* technology yielding a nested profit form with the gross substitutes property. The procedures of McFadden in Chapter II.2 can be used to establish a stronger result. Suppose $\Pi(\mathbf{a},\mathbf{K},\mathbf{p})$ and $\Phi(\mathbf{r},\mathbf{p})$ are the profit functions associated with an arbitrary *ex ante–ex post* technology. Suppose they are differentiable at a point $(\bar{\mathbf{r}},\bar{\mathbf{p}})$, yielding optimal quantities $(\bar{\mathbf{a}},\bar{\mathbf{K}},\bar{\mathbf{x}})$. Suppose that Φ and Π are differentiable at this point, and have the following gross substitute properties:

$$\begin{array}{ll} \partial^2\Pi/\partial a_i\partial a_j \geqq 0, & \partial^2\Pi/\partial K_i\partial K_j \geqq 0, \\ \partial^2\Pi/\partial a_i\partial K_j \geqq 0, & \partial^2\Pi/\partial p_i\partial p_j \leqq 0, \\ \partial^2\Phi/\partial r_i\partial r_j \leqq 0, & \partial^2\Phi/\partial p_i\partial p_j \leqq 0, \\ \partial^2\Phi/\partial p_i\partial r_j \leqq 0. & \end{array}$$

Suppose Π is concave in $\mathbf{a}$, independent of $\mathbf{K}$, and define

$$\Pi(\mathbf{a},\mathbf{K},\mathbf{p}) \doteq \Pi(\bar{\mathbf{a}},\bar{\mathbf{K}},\bar{\mathbf{p}}) + \sum_{l=1}^{L} \alpha_l \Pi_{a_l}(\bar{\mathbf{a}},\bar{\mathbf{K}},\mathbf{p}) + 1/2 \sum_{l,m=1}^{L} \beta_{lm} \Pi_{a_l a_m}(\bar{\mathbf{a}},\bar{\mathbf{K}},\mathbf{p}),$$

where

$$\alpha_l = a_l - \bar{a}_l, \qquad \beta_{lm} = (a_l - \bar{a}_l)(a_m - \bar{a}_m).$$

This function is then linear in the parameters α_l,β_{lm}. Define $\mathbf{B}$ to be the set of parameter vectors $(1,(\alpha_l), (\beta_{lm}))$ corresponding to $(\mathbf{a},\mathbf{K})$ in the domain of Π, and define $\psi(q_0,(q_l), (q_{lm}))$ to be its "profit function". Then the "linearized" Π and the function ψ define a nested profit form that can, by the procedure outlined previously, be approximated to second order by the system (69)–(71). Thus, the tests suggested above should be relatively robust for deviations of the true *ex ante–ex post* production structure from that implied by the linear-in-parameters form. Because the multivariate model estimated in these tests utilizes much more information on the structure of the production process than would parallel non-parametric statistics, it should yield substantially more powerful tests.

8. Concluding Remarks

The preceding chapters in this volume have demonstrated that the theory of duality is a concept that is extremely useful in the estimation of production parameters when production is specified in terms of a single-level decision rule. In this chapter we have extended the application of duality theory to the more complicated two-stage (*ex ante–ex post*) description of technology. This extension allowed us to pursue interesting phenomena, such as the role of uncertainty in the design decision, which cannot be analyzed within the more limited framework.

The main purpose of this chapter was to demonstrate a means of generating functional forms that are capable of empirically describing the *ex ante–ex post* structure. Econometricians who estimate production functions require these functional forms in order to take into account the dynamic efficiency of flexible techniques in a world of durable inputs and uncertain outcomes.