Suggested Solutions to Problem Set 4

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1. Moral hazard and asset-price bubbles.

(a) The representative entrepreneur borrows $B$ on date 1 and invests it in $X_R$ risky assets and $X_S$ safe assets such that

$$B = PX_R + X_S.$$  \hfill (1)

On date 2, the risky and safe assets pay off $RX_R$ and $rX_S$, respectively. In addition, the entrepreneur has to pay the bank $rB$ if $RX_R + rX_S \geq rb$ and $RX_R + rX_S$ otherwise. So, the date 2 payoff to the entrepreneur equals

$$\Pi(R) = \begin{cases} RX_R + rX_S - rB & \text{if } RX_R + rX_S \geq rB \\ 0 & \text{otherwise} \end{cases}.$$  \hfill (2)

Substituting (1),

$$\Pi(R) = \begin{cases} RX_R - rPX_R & \text{if } RX_R - rPX_R \geq 0 \\ 0 & \text{otherwise} \end{cases} = \max \{ RX_R - rPX_R, 0 \}.$$  \hfill (2)

The payoff function $\Pi(R)$ is graphed in figure 1.

(b) The expected payoff $E[\Pi(R)]$ increases as the variance of $R$ rises, given $R$ and $X_R$. The reason is that $\Pi(R)$ is convex. Intuitively, a higher variance increases the probability of values of $R$ closer to 0 and $R_{MAX}$. In the latter case, the payoff to the entrepreneur increases, but in the former case the entrepreneur simply defaults and the payoff is not affected. As a result, the expected payoff increases.
Figure 1: The entrepreneur’s payoff function

(c) Let $U(X_R)$ denote the entrepreneur’s expected net benefits of investing in $X_R$ risky assets. Then, substituting (2) yields

$$U(X_R) = E[\Pi(R)] - c(X_R)$$

$$= \int_0^{rP} 0 \, dH(R) + \int_{rP}^{R_{MAX}} (RX_R - rPX_R) \, dH(R) - c(X_R)$$

$$= \int_{rP}^{R_{MAX}} (RX_R - rPX_R) \, dH(R) - c(X_R).$$

The first-order optimality condition with respect to $X_R$ is

$$U'(X_R) = \int_{rP}^{R_{MAX}} (R - rP) \, dH(R) - c'(X_R) = 0. \quad (3)$$

Rearranging gives

$$c'(X_R) = \int_0^{R_{MAX}} RdH(R) - \int_0^{rP} RdH(R) - \int_{rP}^{R_{MAX}} rPdH(R)$$

$$= \bar{R} - \int_0^{rP} RdH(R) - [1 - H(rP)] rP. \quad (4)$$

(d) Besides the optimality condition (4), there are three other equilibrium conditions. First, equilibrium in asset markets requires that demand
and supply of the risky asset are equal:

\[ X_R = 1. \]  \hspace{1cm} (5)

Second, the entrepreneur’s budget constraint (1) has to hold. Substituting (5) gives

\[ X_S + P = B. \]  \hspace{1cm} (6)

Third, the risk-free gross return is determined by the gross marginal product of the safe production technology at the aggregate safe investment level \( X_S \), so

\[ r = f'(X_S). \]  \hspace{1cm} (7)

We will now show that a unique equilibrium exists when \( \bar{R} > c'(1) \).

Combining (4) and (5),

\[ \bar{R} - c'(1) = \int_0^P R dH(R) + [1 - H(r_P)] r_P. \]  \hspace{1cm} (8)

The right-hand side of this equation is a function of \( r_P \) which we will denote by \( G(r_P) \). Notice that \( G(0) = 0 \) and that \( G(r_P) > 0 \) if and only if \( r_P > 0 \). So, in equilibrium, \( r_P \) is strictly positive when \( \bar{R} > c'(1) \). Furthermore, applying Leibniz’ rule,

\[ G'(r_P) = r_P H'(r_P) + [1 - H(r_P)] - H'(r_P) r_P = 1 - H(r_P) \]

so that \( 0 < G'(r_P) < 1 \) for \( 0 < r_P < R_{MAX} \). Hence, \( G(r_P) \) is a one-to-one function of \( r_P \) and so (8) uniquely determines the value of \( r_P \); to be precise, \( r_P = G^{-1}(\bar{R} - c'(1)) \). Thus, for \( \bar{R} > c'(1) \), (8) defines an inverse relationship between \( r \) and \( P \) which is depicted by the hyperbola RR in figure 2. Combining the remaining two equilibrium conditions, (6) and (7), yields

\[ r = f'(B - P). \]  \hspace{1cm} (9)

Since \( \partial r / \partial P = -f''(B - P) > 0 \), (9) defines a positive relationship between \( r \) and \( P \) which is depicted by the SS schedule in figure 2. The intersection of the RR and SS curves determines the equilibrium values of \( r \) and \( P \). Clearly, there is a unique equilibrium when \( \bar{R} > c'(1) \).

Concerning the banks, they receive \( rB \) if \( RX_R + rX_S \geq rB \), or equivalently, using (5) and (6), if \( R \geq rP \). But if \( R < rP \), they only get
In equilibrium, \( rP > 0 \) and the expected payoff to the banks equals

\[
V = \int_0^{rP} (RX_R + rX_S) dH(R) + \int_{rP}^{R_{MAX}} rBdH(R)
\]

As a consequence, banks earn an expected return strictly below \( r \) on their loans \( B \).

(e) Recall that the downward-sloping RR schedule in figure 2 is defined by (3) and (5), so

\[
\int_{rP}^{R_{MAX}} RdH(R) - Pr\{R \geq rP\} rP - c'(1) = 0.
\]

Rearranging and solving for the asset price gives

\[
P = \frac{1}{r} \left[ \int_{rP}^{R_{MAX}} RdH(R) - c'(1) \right] / Pr\{R \geq rP\}
\]

(f) If entrepreneurs financed asset purchases entirely out of their own wealth \( B \), their payoff would be

\[
\Pi^*(R) = RX_R + rX_S.
\]
Taking into account the opportunity costs \( rB \), the expected net benefits equal

\[
U^*(X_R) = E[\Pi^*(R)] - rB - c(X_R) = \int_0^{R_{MAX}} R X_R dH(R) - rP X_R - c(X_R),
\]

using the budget constraint (6). Then, the first-order optimality condition for the representative entrepreneur is

\[
U'^*(X_R) = \int_0^{R_{MAX}} R dH(R) - rP - c'(X_R) = 0.
\]

Substituting (5) and rearranging gives the fundamental asset price in equilibrium

\[
P^* = \frac{1}{r} \left[ \int_0^{R_{MAX}} R dH(R) - c'(1) \right] = \frac{1}{r} \left[ \bar{R} - c'(1) \right]. \tag{12}
\]

In words, the fundamental price of the risky asset is the present discounted value of the difference between the expected return \( \bar{R} \) and the marginal cost of investment \( c'(1) \). So, \( P^* \) equals the expected present discounted value of the net marginal benefits from risky assets.

\( \text{(g)} \) Rearranging the equilibrium condition for the risky asset (8),

\[
rP = \bar{R} - c'(1) + H(rP) rP - \int_0^{rP} R dH(R) > \bar{R} - c'(1) = rP^*. \tag{13}
\]

The inequality follows from the fact that \( rP > \left[ \int_0^{rP} R dH(R) \right] / H(rP) \), i.e., the conditional expectation of \( R \) from 0 to \( rP \) must be less than \( rP \). The last equality in (13) uses (12) and the assumption that the interest rate \( r \) in equations (11) and (12) is the same. Therefore, \( P > P^* \).

\( \text{(h)} \) In an economy where entrepreneurs finance investment out of their own wealth \( B \), the RR schedule is no longer applicable. Instead, equilibrium in the market for risky assets is described by (12), depicted by the \( R^*R^* \) schedule in figure 3. Since \( P > P^* \) for a given level of the interest rate \( r \), the \( R^*R^* \) locus lies to the lower left of the RR curve. The SS schedule describing the market for safe assets is not affected. So, in the case of self-financed investment, the equilibrium is given by the intersection of the \( R^*R^* \) and SS schedules. Clearly, the equilibrium risk-free interest rate \( r^* \) and asset price \( P^* \) are both lower than their values in the case of bank-financing.
Figure 3: Self-financed versus bank-financed investment

(i) A rise in the amount of bank loans $B$ does not affect the RR schedule. However, the SS locus given by (9) shifts down and to the right, as a higher value for $B$ reduces $r$ for a given level of $P$. The new equilibrium at the intersection of the RR and SS schedules features a lower interest rate $r'$ and a higher asset price $P'$ as illustrated in figure 4. Intuitively, the increased supply of funds $B$ boosts the demand for the risky asset, thereby inflating the asset price $P$, and raises the amount invested in the safe asset, which depresses the risk-free interest rate $r$.

(j) The fact that the loans $B$ are supplied by foreign instead of domestic banks is irrelevant for the derivation of the equilibrium relationships (8) and (9). So, for a given capital inflow $B$, the values of $r$ and $P$ are determined as in part (d). The level of $B$, however, is no longer exogenous; foreign banks lend up to the point where the expected payoff from domestic loans, $V$, equals $r^WB$. Substituting the equilibrium conditions (5) and (6) into (10) we obtain

$$ V = \int_0^{r^P} RdH(R) - H(rP)rP + rB = r^WB. $$
Rearranging gives

$$r = r^W + \frac{1}{B} \left[ H(rP) rP - \int_{0}^{rP} RdH(R) \right].$$

(14)

Since $rP > \left[ \int_{0}^{rP} RdH(R) \right] / H(rP)$, the domestic interest rate exceeds the world interest rate: $r > r^W$.

Formally, we have three equations, (14), (9) and (8), which can be solved for the three endogenous variables $r$, $B$ and $P$. Notice that (14) determines a negative relationship between $r$ and $B$ for a given level of $rP$. This is depicted in figure 5 by the hyperbola FF with horizontal asymptote $r^W$. In addition, we can write (9) as

$$r = f'' \left( B - \frac{(rP)}{r} \right),$$

(15)

where $\frac{\partial r}{\partial B} = -1/[(rP)/r^2 - 1/f''(B - (rP)/r)] < 0$, given $rP$. This also defines a negative relationship between $r$ and $B$ for a given level of $rP$, which is depicted by the SS schedule. The SS curve is convex and bounded below by the hyperbola $r = rP/B$, which is indicated by the dashed line. The open-economy equilibrium is
Figure 5: The effect of an decrease in the world interest rate determined by the intersection of the FF and SS schedules, given the level of \( rP \) determined by (8). A fall in the world interest rate shifts the FF schedule downward as shown in figure 5. As a result, capital inflows \( B \) increase, the domestic interest rate \( r \) falls, and since \( rP \) is not affected, the price of the risky asset \( P \) rises.

2. **Comparing optimal consumption with complete and incomplete markets.**

(a) Ignoring nonnegativity, write the unconstrained maximization as

\[
\max_{B_2} \left[ (1 + r)B_1 - B_2 + Y_1 \right] - \frac{a_0}{2} \left[ (1 + r)B_1 - B_2 + Y_1 \right]^2
\]

\[
+ \frac{1}{1 + r} \mathbb{E}_1 \left\{ \left[ (1 + r)B_2 + Y_2(s) \right] - \frac{a_0}{2} \left[ (1 + r)B_2 + Y_2(s) \right]^2 \right\}.
\]

The first-order condition for \( B_2 \) is:

\[
C_1 = \mathbb{E}_1 \left\{ C_2(s) \right\}.
\]

The \( S + 1 \) budget constraints in the problem imply that

\[
\mathbb{E}_1 \left\{ C_1 + \frac{C_2(s)}{1 + r} \right\} = \mathbb{E}_1 \left\{ (1 + r)B_1 + Y_1 + \frac{Y_2(s)}{1 + r} \right\}.
\]
Thus by substitution of the first-order condition,

\[
\left(1 + \frac{1}{1 + r}\right) C_1 = E_1 \left\{ (1 + r)B_1 + Y_1 + \frac{Y_2(s)}{1 + r} \right\},
\]

that is,

\[
C_1 = \frac{1 + r}{2 + r} E_1 \left\{ (1 + r)B_1 + Y_1 + \frac{Y_2(s)}{1 + r} \right\},
\]

(16)
The implied values of \(C_2(s)\) are given by

\[
C_2(s) = (1 + r)B_2 + Y_2(s).
\]

Substituting for \(B_2 = (1 + r)B_1 + Y_1 - C_1\), where \(C_1\) is given by (16) above, we obtain

\[
C_2(s) = (1+r) \left\{ (1 + r)B_1 + Y_1 - \frac{1 + r}{2 + r} \left( (1 + r)B_1 + Y_1 + \frac{E_1 Y_2}{1 + r} \right) \right\} + Y_2(s).
\]

For the \(\infty\)-horizon case, we just get the usual "permanent-income" formula, essentially eq. (32) of Chapter 2 (suitably adapted).

(b) The consumption formula of part (a) will be generally valid if the nonnegativity constraint on consumption never binds, that is, if, even when output hits its minimal date 2 value (in state \(s = 1\)), \(C_2 \geq 0\). This last inequality will hold if and only if \(C_2(1) \geq 0\), which is equivalent to

\[
(1 + r)B_1 + Y_1 + \frac{2 + r}{1 + r} Y_2(1) \geq E_1 Y_2.
\]

If this inequality does not hold, then the nonnegativity constraint on \(C_2\) binds in at least one state of nature on date 2, so we cannot ignore the associated Kuhn-Tucker multiplier (see supplement A to Chapter 2). In that case, the Kuhn-Tucker theorem predicts that date 1 consumption must be lower than \(E_1 \{ C_2(s) \} \) to make \(C_2(1) = 0\) (in state 1 of date 2 when output is minimal). As a result, the bond Euler equation no longer holds. Furthermore, since

\[
C_2(1) = (1 + r) [(1 + r)B_1 + Y_1 - C_1] + Y_2(1) = 0
\]

we see that \(C_1 = (1 + r)B_1 + Y_1 + Y_2(1)/(1 + r)\).
(c) The state-by-state Euler equations are
\[ \frac{p(s)}{1 + r} (1 - a_0 C_1) = \frac{\pi(s)}{1 + r} [1 - a_0 C_2(s)], \]
which reduce to
\[ C_1 = C_2(s), \forall s, \]
because we’ve assumed \( p(s) = \pi(s) \). Thus consumption is constant across states and dates, equal to \( \bar{C} \), given by
\[ \bar{C} = \left( \frac{1 + r}{2 + r} \right) \left[ (1 + r) B_1 + Y_1 + \sum_{s=1}^{S} \frac{p(s) Y_2(s)}{1 + r} \right], \]
\[ = \left( \frac{1 + r}{2 + r} \right) \left[ (1 + r) B_1 + Y_1 + \sum_{s=1}^{S} \frac{\pi(s) Y_2(s)}{1 + r} \right]. \]
The critical difference between the equation above and eq. (16) is that the preceding equation holds ex post as well as ex ante, i.e., it holds in every state on date 2 as well as on date 1. Equation (16) above, in contrast, implies that date 2 consumption varies one-for-one with the output realization (date 2 consumption is not insured in the “bonds-only” asset regime). Thus the possibility of negative consumption is an issue in the bonds-only case, though not under complete markets.