1 Capital Asset Pricing Model ("Classic" CAPM)

The investor is maximizing the quadratic utility function of his expected wealth in the second period

$$U^i = E\{\alpha W^i - \frac{\gamma}{2}(W^i)^2\},$$

where the expected wealth in the second period is equal to the weighted average of the returns on risky assets and the risk-free asset. The weights are the shares of period one wealth invested in each asset. Since the shares sum up to one, the share of wealth invested in risk-free asset is equal to $1 - \sum_{j=1}^{N} x_j^i$.

(a) This one is easy, just open the brackets and rearrange a little:

$$W^i = W_0^i \sum_{j=1}^{N} x_j^i + \sum_{j=1}^{N} x_j^i r_j + (1 + r_F) - \sum_{j=1}^{N} x_j^i - \sum_{j=1}^{N} x_j^i r_F =$$

$$= W_0^i [(1 + r_F) + \sum_{j=1}^{N} x_j^i (r_j - r_F)],$$

which is what we wanted.

(b) Now the investor is maximizing the utility by choosing the shares of her initial wealth to be invested in each asset. To do that she substitutes the definition of $W^i$ from the result of (a) into the utility function:

$$\max_{x_j^i} U^i = E\{\alpha W^i - \frac{\gamma}{2}(W^i)^2\}$$

$$\max_{x_j^i} E\{\alpha W_0^i[(1 + r_F) + \sum_{j=1}^{N} x_j^i (r_j - r_F)] -$$

$$- \frac{\gamma}{2}(W_0^i)^2[(1 + r_F) + \sum_{j=1}^{N} x_j^i (r_j - r_F)]^2\}.$$
F.O.C. is:

\[ E\{ (r_j - r_F) a W^i_j - \gamma W^i_j (1 + r_F) + \sum_{j=1}^{N} x_j^i (r_j - r_F) \} (r_j - r_F) \} = 0. \]

Note now that \( W^i_j (1 + r_F) + \sum_{j=1}^{N} x_j^i (r_j - r_F) \) in the equation above is just the definition of \( W^i \) so we can simplify the first-order condition as follows:

\[ E\{ a W^i_j (r_j - r_F) - \gamma W^i_j W^i (r_j - r_F) \} = 0, \]

and take everything non-stochastic out of the expectation operator, taking into account that \( EW^i r_j = EW^i Er_j + \text{Cov}(W^i, r_j) \) and that \( r_F \) is a constant.

\[
\begin{align*}
\alpha W^i_0 E(r_j - r_F) &= \gamma W^i_0 E W^i E(r_j - r_F) + \gamma W^i_0 \text{Cov}(W^i, r_j) \\
\alpha E(r_j - r_F) &= \gamma EW^i E(r_j - r_F) + \gamma \text{Cov}(W^i, r_j).
\end{align*}
\]

We can now express what we are actually interested in -- the expected risk premium on asset \( j \)

\[ E(r_j - r_F) = \frac{\gamma \text{Cov}(W^i, r_j)}{\alpha - \gamma EW^i}. \]

And for the point (c) we can again rewrite it as

\[ E(r_j - r_F) \alpha - \gamma E(r_j - r_F) EW^i = \gamma \text{Cov}(W^i, r_j). \]

(c) Now we can sum the last equation over \( i \). Recall from whatever probability you have learnt or convince yourself using the definition of covariance that \( \text{Cov}(\sum_i X_i, Y) = \sum_i \text{Cov}(X_i, Y) \). This allows us to find that

\[ ME(r_j - r_F) \alpha - \gamma E(r_j - r_F) EW = \gamma \text{Cov}(W^i, r_j), \]

or

\[ E(r_j - r_F) = \frac{\gamma \text{Cov}(W^i, r_j)}{M \alpha - \gamma EW^i}. \] (1)

2
(d) For each individual the Arrow-Pratt coefficient of relative risk aversion \( \rho^i \) is

\[
\rho^i = \frac{U''(W^i)\text{EW}^i}{U'(W^i)} = \frac{\gamma \text{EW}^i}{a - \gamma \text{EW}^i}.
\]

If instead of \( W^i \) we use average wealth \( \frac{W}{\bar{W}} \), we will get the “average” coefficient of relative risk aversion:

\[
\rho = \frac{\gamma W}{a - \gamma \frac{W}{\bar{W}}}
\]

(e) Let’s multiply the numerator and the denominator of the equation (1) in part (c) by \( \frac{W}{\bar{W}} \). From the definition of \( r_M \) we know that \( \frac{E_W}{\bar{W}} = E(1 + r_M) \). So we get

\[
E(r_j - r_F) = \frac{\frac{E_W}{\bar{W}} \text{Cov}(W, r_j)}{(a - \gamma \frac{W}{\bar{W}})E(1 + r_M)} = \frac{\gamma \frac{E_W}{\bar{W}} - \text{Cov}(W, r_j)}{(a - \gamma \frac{E_W}{\bar{W}})E(1 + r_M)}.
\]

Another useful property of covariance is \( c \text{Cov}(X, Y) = \text{Cov}(cX, Y) \), where \( c \) is a constant. Since \( W_0 \) is a constant and given the definition of \( \rho \) in (d), we derive

\[
E(r_j - r_F) = \frac{\rho \text{Cov}(\frac{W}{\bar{W}}, r_j)}{E(1 + r_M)},
\]

or

\[
E(r_j - r_F) = \frac{\rho \text{Cov}(r_M, r_j)}{E(1 + r_M)}.
\]

(f) The expected risk premium on asset \( j \) is positive if the return on asset \( j \) is positively correlated with a market return, because such an asset does not provide insurance against the bad state of nature, instead, the return on it is low in the bad state of nature and therefore it is very risky. If asset \( j \) is negatively correlated with the market, then the expected risk premium is negative - investors want to buy this asset for insurance reasons and not for the high return. The risk-free asset is not correlated with the market and thus is

\footnote{see Obstfeld and Rogoff page 278}
not providing any insurance, so the risk premium is zero. Basically all this is a consumption smoothing idea.

Not surprisingly, the expected risk premium (in absolute value) positively depends on $\rho$. The more risk averse are the investors the higher expected risk premium they request on the assets positively correlated with market and the more they are ready to give up in order to get a negatively correlated with market asset.

For positively correlated with market assets the expected risk premium will increase if the expected market return increases, which is just the corrolary of the fact that they are positively correlated. The opposite holds for the negatively correlated with market assets.

(g) Equation (2) above must hold for the “market”, because “market” is also a risky asset:

$$E(r_M - r_F) = \frac{\rho \text{Cov}(r_M, r_M)}{E(1 + r_M)}. \quad (3)$$

Now divide (2) by (3) to get

$$\frac{E(r_j - r_F)}{E(r_M - r_F)} = \frac{\text{Cov}(r_j, r_M)}{\text{Var}(r_M)},$$

$$E(r_j - r_F) = \frac{\text{Cov}(r_j, r_M)}{\text{Var}(r_M)} E(r_M - r_F) =$$

$$= \beta_j E(r_M - r_F)$$

using the definition of $\beta$. This is CAPM.

2 Lucas’s CCAPM

The only output is perishable fruits from the tree. In equilibrium, total economy’s consumption equals total output, since fruits can not be stored.

(a) The condition we are supposed to derive is nothing other then Euler equation.

Formal way:

$$\max_{c_t, x_t} \{ \sum_{s=t}^{\infty} e^{-\theta(s-t)} u(c_s) \}$$

s.t.

$$c_t + p_t x_{t+1} \leq (y_t + p_t) x_t.$$
Note that maximization implies that the constraint will be binding, otherwise consumption can be increased. We will use dynamic programming described in the Supplement A.2 to chapter 2.

We will maximize over $x$ the value function

$$V_t(x_t) = \max_{c_t} \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} e^{-\delta(s-t)} u(c_s) \right\}$$

using the Bellman equation

$$V_t(x_t) = \max_{c_t} \left\{ u(c_t) + e^{-\delta} \mathbb{E}_t \{ V_{t+1}(x_{t+1}) \} \right\},$$

where from the budget constraint

$$x_{t+1} = \left( y_t + p_t \right) x_t - c_t$$

and thus

$$\frac{dV_{t+1}(x_{t+1})}{dc_t} = -\frac{1}{p_t} V'(x_{t+1}).$$

F.O.C:

$$u'(c_t) - e^{-\delta} E_t \{ V'(x_{t+1}) \} \frac{1}{p_t} = 0.$$

We now can use the envelope theorem to derive $V_{t+1}'(x_{t+1})$:

$$V_{t+1}'(x_{t+1}) = u'(c_{t+1}) \frac{\partial c_{t+1}}{\partial x_{t+1}}.$$

By rewriting the binding budget constraint one period forward we get $c_{t+1} = (y_{t+1} + p_{t+1}) x_{t+1} - p_{t+1} x_{t+2}$ and $\frac{\partial c_{t+1}}{\partial x_{t+1}} = y_{t+1} + p_{t+1}$ and thus

$$V_{t+1}'(x_{t+1}) = u'(c_{t+1}) (y_{t+1} + p_{t+1}).$$

If we substitute the last equation into Bellman equation we get

$$u'(c_t) - e^{-\delta} E_t \{ u'(c_{t+1}) (y_{t+1} + p_{t+1}) \} \frac{1}{p_t} = 0$$

or

$$p_t u'(c_t) = e^{-\delta} E_t \{ (p_{t+1} + y_{t+1}) u'(c_{t+1}) \},$$

which is the Euler equation we were supposed to derive.

\footnote{see page 718 in Obstfeld and Rogoff}
Intuitive way:
Let us rewrite this Euler equation as
\[ u'(c_t) = e^{-\delta} E_t \{ \left( \frac{P_{t+1}}{P_t} + \frac{y_{t+1}}{y_t} \right) (u'(c_{t+1})) \}. \]

To buy one more (infinitesimal) share of tree consumer should give up \( u'(c_t) \) of his consumption today. He expects this share to bring dividends \( \frac{y_{t+1}}{y_t} \) and capital gains \( \frac{P_{t+1}}{P_t} \) tomorrow. If the consumer is maximizing his utility, he would equalize today’s marginal cost of buying the share (LHS) to the discounted value of the tomorrow expected benefit of having this share (RHS).

(b) In equilibrium, as mentioned above, total consumption is equal to total output and, since all individuals are identical
\[ p_t = e^{-\delta} E_t \{ (P_{t+1} + y_{t+1}) \frac{u'(y_{t+1})}{u'(y_t)} \}. \]

Iterating this equation forward and substituting the results for \( P_{t+s} \), we get
\[ p_t = E_t \left[ e^{-\delta} \frac{u'(y_{t+1})}{u'(y_t)} + e^{-\delta} \frac{u'(y_{t+2})}{u'(y_t)} + \ldots + e^{\delta(T-t)} \frac{u'(y_{T+T})}{u'(y_T)} \right]. \]

Taking the limit as \( T \to \infty \):
\[ p_t = E_t \left[ \sum_{s=t+1}^{\infty} e^{-\delta(s-t)} y_s \frac{u'(y_s)}{u'(y_t)} + E_t \left[ \lim_{T \to \infty} e^{-\delta(T-t)} \frac{u'(y_T)P_{T+t}^*}{u'(y_T)} \right] \right]. \]

If we impose no-bubble condition \( E_t \lim_{T \to \infty} e^{-\delta(T-t)} P_{T+t}^* = 0 \) we have the “fundamentals” price that only depends on dividends:
\[ p_T^* = E_t \left[ \sum_{s=t+1}^{\infty} e^{-\delta(s-t)} y_s \frac{u'(y_s)}{u'(y_t)} \right]. \]

(c) We can rewrite the last equation from (b) as
\[ p_T^* = \sum_{s=t+1}^{\infty} e^{-\delta(s-t)} \left[ E_T \left( \frac{u'(y_s)}{u'(y_t)} E_T(y_s) + \text{Cov} \left( \frac{u'(y_s)}{u'(y_t)}, y_s \right) \right) \right]. \]

The first term on the RHS is the PDV of all future dividends (the expected payoff), the second is the “risk premium”. The risk premium is negative since the return on asset is positively (in fact, perfectly positively) correlated with
consumption and as usual we assume $u'' < 0$ and therefore marginal utility is a decreasing function of consumption. Intuition is the same as for CAPM (problem 1) - the consumers will pay less for the share of the tree, since its return is low on the “bad day” and is not providing any insurance. In contrary, by buying the additional share consumers accept additional risk and therefore price must be lower to encourage them to buy this additional share. \(^3\)

(d) We do not really need the specific utility function for this question, we’ll use it later in (e). From the beginning of the question we know:

$$\log y_t = \log y_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

thus

$$\log y_s = \log y_t + \sum_{j=1}^{t-s} \epsilon_{t+j} = \log y_t + \epsilon_t, \quad \text{where } \epsilon_t \sim \mathcal{N}(0, \sum_{j=1}^{t-s} \sigma^2),$$

since $\epsilon_t$ is i.i.d.\(^4\)

$$(1 - \rho) \log y_s = (1 - \rho) \log y_t + (1 - \rho) \epsilon_t,$$

$$\text{Var}(1 - \rho) \log y_t) = (1 - \rho)^2 \text{Var}(\log y_t),$$

$$E((1 - \rho) \log y_s) = (1 - \rho) \log y_t,$$

therefore $y_s^{1-\rho} \sim \lognormal((1 - \rho) \log y_t, (1 - \rho)^2 (s - t) \sigma^2)$ and using properties of lognormal distribution, we get

$$E y_s^{1-\rho} = e^{(1-\rho) \log y_t + (1-\rho)^2 (s - t) \frac{\sigma^2}{2}} =$$

$$y_t^{1-\rho} e^{(1-\rho)^2 (s - t) \frac{\sigma^2}{2}}$$

which is what we were supposed to derive.

(e) For a given in (d) utility function $u'(c) = e^{-\theta c}$. Substitute it to the equation for $p_t^*$ and use the result in (d) to get

$$p_t^* = E_t \left[ \sum_{s=t+1}^\infty e^{-\theta (s-t)} \frac{y_s^{1-\rho} y_s}{y_t^{1-\rho}} \right]$$

\(^3\)High-low discussion here seems to contradict the discussion in problem 1. Actually, it does not, it is exactly the same logic. The difference is that in problem 1 we were talking about return on asset and here we are talking about the price of an asset and these two are negatively correlated.

\(^4\)note also that $\sum_{j=1}^{t-s} \sigma^2 = (s - t) \sigma^2$. 
\[ \sum_{s=t+1}^{\infty} e^{-\theta(s-t)} \frac{y_{t+1} e^{\frac{\sigma^2(1-\rho)^2}{2}(s-t)}}{y_t^{\theta}} = \]
\[ = \frac{y_t}{y_{t+1}} \sum_{s=t+1}^{\infty} e^{-\theta + \frac{\sigma^2(1-\rho)^2}{2}(s-t)} \cdot \]

Note that the last sum converges if \( -\theta + \frac{\sigma^2(1-\rho)^2}{2} < 0 \) or \( \theta > \frac{\sigma^2(1-\rho)^2}{2} \), otherwise it explodes. Imposing this condition we can then write
\[ p_t^* = y_t \frac{e^{\frac{\sigma^2(1-\rho)^2}{2} - \theta}}{1 - e^{\frac{\sigma^2(1-\rho)^2}{2} - \theta}} = \frac{y_t}{1 - e^{\frac{\sigma^2(1-\rho)^2}{2} - \theta}} = \chi y_t, \]
where \( \chi = \frac{1}{e^{\frac{\sigma^2(1-\rho)^2}{2} - \theta}} - 1 \).

(f) Let’s rewrite the Euler equation derived in (a) replacing \( p \) with \( p^* + b \) everywhere:
\[ (p_t^* + b_t) u'(y_t) = e^{-\theta} E_t \{ (p_{t+1}^* + b_{t+1} + y_{t+1}) u'(y_{t+1}) \}. \]

or
\[ p_t^* u'(y_t) + b_t u'(y_t) = e^{-\theta} E_t \{ (p_{t+1}^* + y_{t+1}) u'(y_{t+1}) \} + e^{-\theta} E_t \{ b_{t+1} u'(y_{t+1}) \}. \]

As \( p^* \) is the “fundamentals” price, it is satisfies the Euler equation and we only have to check that “bubble” satisfies it too:
\[ b_t u'(y_t) = e^{-\theta} E_t \{ b_{t+1} u'(y_{t+1}) \}, \]
now substitute \( b_t = \frac{A y_t^\lambda}{u'(y_t)} \) and similar for \( b_{t+1} \)
\[ A y_t^\lambda = e^{-\theta} E_t \{ A y_{t+1}^\lambda \}. \]

Using the formula for the expected value of the lognormal distribution we can write \( E_t(y_{t+1}^\lambda) = y_t^\lambda e^{\lambda^2 \sigma^2/2} \). Or, taking into account that \( \lambda^2 = \frac{\sigma^2}{\rho^2} \),
\[ A y_t^\lambda = e^{-\theta} \{ A y_{t+1}^\lambda e^{\frac{\sigma^2}{\rho^2} \sigma^2} \}, \]
\[ A y_t^\lambda = A y_{t+1}^\lambda, \]
i.e. Euler equation holds for the price with a bubble.

Now let’s check the transversality condition:
\[ \lim_{T \to \infty} e^{-\theta(T-t)} E_t \{ u'(y_{t+T}) (p_{t+T} + b_{t+T}) \} = \lim_{T \to \infty} e^{-\theta(T-t)} E_t \{ u'(y_{t+T}) p_{t+T} \} + \lim_{T \to \infty} e^{-\theta(T-t)} E_t \{ u'(y_{t+T}) b_{t+T} \}. \]
of which the first limit is zero because $p^*$ is the “fundamentals” price. The second limit is equal to:

$$
\lim_{T \to \infty} e^{-\delta(T-t)} E_t \{ A y_t^r \} = \\
= \lim_{T \to \infty} e^{-\delta(T-t)} A y_t^r e^{\frac{\delta}{\sigma^2} \frac{\sigma^2}{2} T} = e^{\delta t} A y_t^r \neq 0,
$$

i.e. the price with a bubble does not satisfy the transversality condition.

(g) In part (b) we derived:

$$
p_t u'(y_t) = E_t \{ \sum_{s=t+1}^{\infty} e^{-\delta(s-t)} y_s u'(y_s) \} + \lim_{T \to \infty} e^{-\delta(T-t)} E_t \{ u'(y_T | p_T + T) \}.
$$

The second term above can not be strictly negative in equilibrium: the only negative term there can be $u'$ and if $u' < 0$ individuals can be better off by throwing away some consumption (because of free disposal). On the other hand this limit can not be strictly positive in equilibrium either: the positive limit implies that agents are accumulating wealth without ever consuming it, so they can be better off by decreasing wealth and increasing consumption. This possibility is not ruled out by the Euler equation because the Euler equation gives the optimality condition between two periods and does not put any restrictions on the overall consumption path.

3 Cagan’s model

(a) As on the lecture, agents are maximizing

$$
\int_t^{\infty} e^{-\delta(s-t)} \{ u(c(s)) + v(m(s)) \} ds
$$

s.t.

$$
\dot{a} = y + ra + h - im - c, \quad a(0) \text{ is given.}
$$

The present value Hamiltonian is thus:

$$
H = [u(c(s)) + v(m(s))] e^{-\delta(s-t)} + \lambda e^{-\delta(s-t)} (y + ra + h - im - c),
$$

with control variables $c$ and $m$ and the state variable $a$. The F.O.C.s are:

$$
\frac{\partial H}{\partial c} = 0: \quad u'(c(s)) = \lambda, \quad (4) \\
\frac{\partial H}{\partial m} = 0: \quad v'(m(s)) = \lambda i, \quad (5) \\
-\frac{\partial H}{\partial a} = \frac{d}{ds} (\lambda e^{-\delta(s-t)}): \quad \dot{\lambda} = \lambda (\delta - r). \quad (6)
$$
To derive money demand divide (5) by (4) and use the fact that the utility of money in our case is $v(m) = \frac{m^\gamma}{\gamma}$ to get

$$\frac{m^{\gamma-1}}{u'(c)} = i,$$

$$(\gamma - 1) \ln m = \ln i + \ln u'(c),$$

$$\ln m = \frac{\ln i + \ln u'(c)}{\gamma - 1}. $$

The elasticity of the money demand

$$-\frac{d \ln m}{d \ln i} = \frac{1}{1 - \gamma}. $$

Note for future reference that this elasticity is greater then 1 iff $\gamma > 0$.

(b) Remember from the lecture that by definition of $m$ and using equilibrium conditions $\delta = r$ and $c = y - g$ as well as the money demand equation, we can write $\frac{\mu}{m} = \mu - \pi = \mu + \delta - i = \mu + \delta - \frac{m^{\gamma-1}}{u'(y)}$, which gives us the equation of motion for $m$:

$$m = [\mu + \delta - \frac{m^{\gamma-1}}{u'(y)}]m,$$

and we will have the “speculative hyperinflation” as in Case 1 in class iff

$$\lim_{m \to 0} m^{\gamma-1}m = 0,$$

$$\lim_{m \to 0} m^{\gamma} = 0,$$

which holds only for $\gamma > 0$ and thus requires the elasticity of money demand to be greater than 1.

(c) Continue to assume $\gamma > 0$. Now

$$v(m) = \frac{m}{\gamma}[1 - \ln \frac{m}{\kappa}],$$

and

$$v'(m) = \frac{1}{\gamma} - \frac{1}{\gamma} \ln \frac{m}{\kappa} - \frac{1}{\gamma} = -\frac{1}{\gamma} \ln \frac{m}{\kappa},$$

which is positive iff $m < \kappa$;

$$v''(m) = -\frac{1}{\gamma m} < 0.$$

\footnote{Since we assume that money supply is constant, the hyperinflation condition $P \to \infty$ implies that $m \to 0$.}
Since we normalized \( u'(y) = 1 \) our money demand is given by \( v'(m) = i \) and we can write:

\[
-\frac{1}{\gamma} \ln \frac{m}{\kappa} = i,
\]

\[
\ln \frac{m}{\kappa} = -\gamma i,
\]

\[
\frac{m}{\kappa} = e^{-\gamma i},
\]

\[
m = \kappa e^{-\gamma i},
\]

which is the Cagan equation. We derived it as a result of utility maximization.

(d) We can rewrite our formula for elasticity as \( \frac{dm}{di} \), then

\[
\frac{dm}{di} = \kappa (-\gamma) e^{-\gamma i},
\]

\[
\frac{dm}{di} \frac{i}{m} = \frac{\gamma \kappa e^{-\gamma i} i}{\kappa e^{-\gamma i}} = \gamma i \xrightarrow{i \to \infty} \infty.
\]

(e) Using the results in (c)

\[
\lim_{m \to 0} m v'(m) = \lim_{m \to 0} -\frac{1}{\gamma} \ln \frac{m}{\kappa} = \lim_{m \to 0} \frac{\ln \frac{m}{\kappa}}{m} = \frac{\frac{1}{m} \ln \frac{m}{\kappa}}{m}.
\]

Note that in the last expression both the numerator and the denominator are going to infinity as \( m \to 0 \) and thus we can use the l'Hospital’s Rule:\(^6\)

\[
\lim_{m \to 0} \frac{\ln \frac{m}{\kappa}}{m} = \lim_{m \to 0} \frac{\frac{1}{m} \frac{m}{\kappa}}{m} = 0,
\]

and the speculative hyperinflation is possible.

(f) Continue to assume that \( \gamma > 0 \). Combining the Cagan equation and the Fisher equation and using the equilibrium fact that \( \delta = r \) we derive

\[
m = \kappa e^{-\gamma (\delta + \pi)}
\]

and

\[
\dot{m} = \kappa \dot{e}^{-\gamma (\delta + \pi) (-\gamma) \pi}
\]

which implies

\[
\frac{\dot{m}}{m} = -\gamma \pi.
\]

From the government budget constraint \( \mu = \frac{g}{m} \) and thus the equilibrium condition is

\[
\pi = \frac{g}{m} - \frac{\dot{m}}{m} = \frac{g}{m} + \gamma \pi
\]

\(^6\)i.e. the limit will not change if we take the derivatives of the numerator and the denominator separately.
from which
\[ \dot{\pi} = \frac{g}{\gamma m} + \frac{\pi}{\gamma} = -\frac{g}{\gamma \kappa e^{-\gamma(\delta + \tau)}} + \frac{\pi}{\gamma} = -\frac{g}{\gamma \kappa} e^{\gamma(\delta + \tau)} + \frac{\pi}{\gamma}. \]

The steady state is when \( \dot{\pi} = 0 \), i.e. when
\[ \frac{g}{\gamma \kappa} e^{\gamma(\delta + \tau)} = \frac{\pi}{\gamma}. \]

This last equation has either no solutions or two solutions (for small \( \gamma \)'s). You can convince yourself in it by plotting RHS and LHS of the equation on the same graph. Therefore we have two steady states. Let’s call them \( \pi_1 \) and \( \pi_2 \).

Also note from the equation of motion of \( \pi \) that when \( \pi = 0 \), \( \dot{\pi} = -\frac{2}{\gamma \kappa} \). And that \( \dot{\pi} \to -\infty \) as \( \pi \to \infty \). So we can draw the phase diagram (see picture) from which we can see that steady state \( \pi_1 \) is unstable and high-inflation steady state \( \pi_2 \) is stable.