1. Give simple examples of discrete-time stochastic processes which are:

(a) white noise but not i.i.d.;
(b) weakly but not strongly stationary;
(c) strongly stationary but not weakly ergodic;
(d) weakly ergodic but not weakly stationary.

2. A stochastic process $x_t$ is bounded if $\Pr\{|x_t| \leq M\} = 1$ for some $M$. A linear filter $a(L)$ is stable if $y_t = a(L)x_t$ is bounded whenever $x_t$ is bounded. Show that absolute summability of the coefficients of the lag polynomial $a(L)$ implies that it is stable.

3. If a $p$-order autoregressive process $\phi(L)y_t = \varepsilon_t$ is stationary, with moving average representation $y_t = \psi(L)\varepsilon_t$, show that

$$0 = \sum_{j=0}^{p} \phi_j \psi_{k-j}$$

$$= \phi(L)\psi_k, \quad k = 0, 1, ...,$$

i.e., show that the moving average coefficients satisfy the autoregressive difference equation.

4. Determine the orders of $p$ and $q$ and find the coefficients of the simplest $ARMA(p, q)$ process which will generate the following autocovariance sequence:

$$\gamma(s) = \begin{cases} 2 & \text{if } s = 0 \\ 1.5(0.5)^{|s|-1} & \text{if } s \neq 0 \end{cases}.$$ 

5. For the specific $ARMA(1, 1)$ process

$$y_t = 0.9y_{t-1} + \varepsilon_t + 0.5\varepsilon_{t-1},$$

where $\varepsilon_t$ is white noise with variance $\sigma^2 = 2$, suppose it is known that $P_T(y_{T+1}) = 1$. Calculate the minimum MSE linear projection of $S \equiv y_{T+1} + y_{T+2} + y_{T+3} + y_{T+4}$ on $\{y_t, y_{t-1}, ..., \}$.

6. Consider the following $ARMA(2, 2)$ process:

$$y_t = y_{t-1} - 0.29y_{t-2} + \varepsilon_t + 3\varepsilon_{t-1} - 10\varepsilon_{t-2},$$

where $\varepsilon_t$ is a white noise process with variance equal to one.

(a) Show that the $y_t$ process is stationary, but that the moving average representation above is not invertible.
(b) Find an alternative ARMA representation for $y_t$ which is both stationary and invertible.

(c) Given $y_T = y_{T-1} = 1$ and $P_{T-1}[y_T] = P_{T-2}[y_{T-1}] = 1$, what is $P_{T+1}[y_T]$? How does your answer change if $P_{T-1}[y_T] = P_{T-2}[y_{T-1}] = 0.9$?

7. Consider the $ARIMA(0,1,1)$ model

$$y_t = y_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1},$$

where the $\{\varepsilon_t\}$ are unobserable white noise innovations with variance $\sigma^2$. Show that the best forecast of $y_T$ given the past history $y_{T-1}, y_{T-2}, \ldots$, can be expressed as an exponentially-weighted average of the lagged values. What is the one-step ahead forecast error variance? How does this model behave as the moving average component approaches non-invertibility?