Conditions for Stationarity and Invertibility

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Linear Processes and the Wold Decomposition

If we permit the order q of a MA(q) process to increase to infinity – that is, if we write

$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

with $\varepsilon_t \sim WN(\sigma^2)$ and $\psi_0 \equiv 1$, we obtain what is known as a *linearly indeterministic* process, denoted $y_t \sim MA(\infty)$. This process is well-defined (in a mean-squared error sense) if the sequence of moving average coefficients $\{\psi_s\}$ is *square-summable*,

$$\sum_{j=0}^{\infty} \psi_j^2 < \infty,$$

in which case it is easy to see that

$$\gamma_y(0) = Var(y_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2,$$

and, more generally,

$$\gamma_y(s) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+s}.$$

By recursion, stationary ARMA processes can be written as linearly deterministic processes; for example, a stationary AR(1) process $y_t = \alpha + \beta y_{t-1} + \varepsilon_t$ has $\theta_s \equiv \beta^s$. Conversely, the MA coefficients for any linearly indeterministic process can be arbitrarily closely approximated by the corresponding coefficients of a suitable ARMA process of sufficiently high order.

Wold showed that *all* covariance stationary stochastic processes could be decomposed as the sum of deterministic and linearly indeterministic processes which were uncorrelated at all leads and lags; that is, if y_t is covariance stationary, then

$$y_t = x_t + z_t,$$

where x_t is a covariance stationary deterministic process (as defined above) and z_t is linearly indeterministic, with $Cov(x_t, z_s) = 0$ for all t and s. This result gives a theoretical underpinning to Box and Jenkins' proposal to model (seasonally-adjusted) scalar covariance stationary processes as ARMA processes.

A linearly indeterministic process y_t is said to be a *generalized linear* process if the white noise components $\{\varepsilon_t\}$ are independently and identically distributed over t; it is said to be a *linear process* if it satisfies the additional restriction that the moving average coefficients are *absolutely summable*, i.e.,

$$\sum_{j=0}^{\infty} |\psi_j| < \infty.$$

Since the square-summability condition implies $\theta_j \to 0$ as $j \to \infty$, absolute summability is a stronger requirement than square summability. This stronger condition implies absolute summability of the autocovariance function $\gamma_y(s)$, since

$$\begin{aligned}
\sum_{s=-\infty}^{\infty} |\gamma_y(s)| &= \left\{ 2\sum_{s=0}^{\infty} |\gamma_y(s)| \right\} - |\gamma(0)| \\
&\leq 2\sigma^2 \sum_{s=0}^{\infty} \left| \sum_{j=0}^{\infty} \psi_j \psi_{j+s} \right| \\
&\leq 2\sigma^2 \sum_{j=0}^{\infty} |\psi_j| \sum_{s=0}^{\infty} |\psi_{j+s}| \\
&\leq 2\sigma^2 \sum_{j=0}^{\infty} |\psi_j| \sum_{s=0}^{\infty} |\psi_s| \\
&= 2\sigma^2 \left(\sum_{j=0}^{\infty} |\psi_j| \right)^2 .
\end{aligned}$$

And when the autocovariance sequence is absolutely summable, then it is summable, so that

$$\begin{aligned}
V_0 &= \lim_{T\to\infty} Var(\sqrt{T}\bar{y}_T) \\
&= \sum_{s=-\infty}^{\infty} \gamma_y(s) \\
&< \infty.
\end{aligned}$$

Thus the stronger requirement that the y_t process is linear (with absolutely summable MA coefficients) is often imposed to ensure applicability of a central limit theorem for $\bar{y}_T$.

Invertibility of Moving Average Processes

If an $MA(q)$ process

$$\begin{aligned}
y_t &= \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q} \\
&= \mu + \theta(L)\varepsilon_t
\end{aligned}$$

can be rewritten as a linear combination of its past values $\{y_{t-s},\ s = 1, 2, ...\}$ plus the contemporaneous error term ε_t, i.e.,

$$y_t = \alpha + \sum_{s=1}^{\infty} \pi_s y_{t-s} + \varepsilon_t$$

for some α and $\{\pi_j\}$, then the process is said to be *invertible.* Consider, for example, the case of $MA(1)$ with $\mu = 0$,

$$y_t = \varepsilon_t + \theta\varepsilon_{t-1},$$

which can be rewritten as

$$\varepsilon_t = y_t - \theta\varepsilon_{t-1}.$$

Repeated substitution of this relation for the lagged ε_{t-s} terms yields

$$\begin{aligned}
\varepsilon_t &= y_t - \theta\left(y_{t-1} - \theta\varepsilon_{t-2}\right) \\
&= y_t - \theta y_{t-1} + \theta^2\varepsilon_{t-2} \\
&\quad ... \\
&= y_t - \theta y_{t-1} + ... + (-\theta)^p y_{t-p} + (-\theta)^{p+1}\varepsilon_{t-p+1}.
\end{aligned}$$

If $|\theta| < 1$, then the last term in this expression tends to zero in mean-square as $p \to \infty$, so that it make sense to write

$$\varepsilon_t = y_t + \sum_{s=1}^{\infty} (-\theta)^s y_{t-s},$$

or

$$y_t = \varepsilon_t + \sum_{s=1}^{\infty} (-\theta)^s y_{t-s},$$

so $|\theta| < 1$ is the sufficient condition for a $MA(1)$ process to be invertible.

Strictly speaking, this is the condition for the process to be "invertible in the past" – when $|\theta| > 1$, a similar recursive argument yields

$$y_t = \theta^{-1}\varepsilon_{t-1} - \sum_{s=1}^{\infty} (-\theta^{-1})^{s-1} y_{t+s},$$

so that y_t is a white-noise error term plus a one-sided linear filter in *future* values. Alas, this expression isn't of much use for forecasting. However, as long as $|\theta| \neq 1$, we can always rewrite a noninvertible MA process as an invertible one

The condition for invertibility of a $MA(1)$ process is the counterpart to the condition of stationarity of an $AR(1)$ process; if

$$y_t = \beta y_{t-1} + \varepsilon_t,$$

then $|\beta| < 1$ implies

$$y_t = \varepsilon_t + \sum_{s=1}^{\infty} \beta^s \varepsilon_{t-s},$$

a $MA(\infty)$ representation with coefficients $\psi_s = \beta^s$. More generally, invertibility of an $MA(q)$ process is the flip side of stationarity of an $AR(p)$ process; that is, an AR process

$$\phi(L)y_t = \alpha + \varepsilon_t$$

is stationary if it can be written as

$$y_t = \mu + \psi(L)\varepsilon_t,$$

where $\psi(L)$ is a one-sided (possibly infinite-order) lag polynomial with square-summable coefficient, while a MA process

$$y_t = \mu + \theta(L)\varepsilon_t$$

is invertible if it can be written as

$$\tau(L)y_t = \alpha + \varepsilon_t,$$

again with a one-sided lag polynomial $\tau(L) \equiv 1 - \pi(L)L$ of (possibly) infinite order.

Invertibility of Lag Polynomials

The general condition for invertibility of $MA(q)$ involves the *associated polynomial equation* (or APE),

$$\begin{aligned}\tilde{\theta}(z) &= z^q \theta(z^{-1}) \\ &= z^q + \theta_1 z^{q-1} + ... + \theta_{q-1} z + \theta_q;\end{aligned}$$

if the (real or complex) solutions $\{z_j^*, j = 1, ..., q\}$ of the polynomial equation $\tilde{\theta}(z^*) = 0$ are inside the unit circle – $|z^*| < 1$ – then the moving average polynomial is invertible. “Invertibility" here means that the rational function $1/\theta(L)$ has a convergent series expression in powers of L,

$$\frac{1}{\theta(L)} = \pi(L),$$

just as stationarity of an $AR(p)$ process means $1/\phi(L)$ has a convergent series expression in powers of L. The rationale for the invertibility condition comes from the Fundamental Theorem of Algebra, which states that any homogeneous p^{th} order polynomial, such as the APE, can be factored into a product of first-order polynomials involving the roots $\{z_j^*\}$:

$$\tilde{\theta}(z) = \prod_{j=1}^{q}(z - z_j^*).$$

Given this expression for $\tilde{\theta}(z)$, we can derive a corresponding representation for the MA polynomial $\theta(L)$ using the inverse relation

$$\theta(z) = z^q \tilde{\theta}(z^{-1}),$$

so that

$$\begin{aligned}
\theta(L) &= L^q \prod_{j=1}^{q}(L^{-1} - z_j^*) \\
&= \prod_{j=1}^{q}(1 - z_j^* L),
\end{aligned}$$

a product of first-order linear filters with coefficients $\{z_j^*\}$. If all the roots $\{z_j^*\}$ were real, then the condition $|z_j^*|$ is just the same condition as $|\theta| < 1$ for a first-order MA process. Even if some of the roots are complex, the Fundamental Theorem of Algebra states that any complex roots of the polynomial appear in conjugate pairs, i.e., if

$$z_j^* = a + bi$$

for some nonzero real number b, then for some other index k,

$$z_k^* = a - bi.$$

Multiplying $(1 - z_j^* L)$ and $(1 - z_k^* L)$ together yields

$$\begin{aligned}
(1 - z_j^* L)(1 - z_k^* L) &= 1 - (z_j^* + z_k^*)L + z_j^* z_k^* L^2 \\
&= 1 - 2aL + (a^2 + b^2)L^2,
\end{aligned}$$

a second-order lag polynomial with real coefficients. So any finite-order lag polynomial can be written as a product of first- and/or second-order lag polynomials with real coefficients.

The condition for invertibility of the lag polynomial $\theta(L)$ (or of $\phi(L)$) can be written in terms of the roots of the lag polynomial itself. Since $\theta(z) = z^q \tilde{\theta}(z^{-1})$, the roots of $\theta(z)$ and $\tilde{\theta}(z)$ are inversely related: that is, if $\lambda \neq 0$ has

$$\theta(\lambda) = 0,$$

then it must be true that $z^* = \lambda^{-1}$ has $\tilde{\theta}(z^*) = 0$, i.e.,

$$\tilde{\theta}(\lambda^{-1}) = 0.$$

So the lag polynomial $\theta(L)$ is invertible if the condition $\theta(\lambda) = 0$ implies that the root λ is *outside the unit circle*, i.e., $|\lambda| > 1$.