Handout #1

Derivation of Factor Price Frontier Expression for Two-Sector Incidence Model

By definition of the elasticities of substitution in production, we know that

\[ \dot{K}_X - \dot{L}_X = \sigma_X (\dot{w} - \dot{r}) \quad \text{and} \quad \dot{K}_Y - \dot{L}_Y = \sigma_Y (\dot{w} - \dot{r}) \]

For convenience, express \( K \) and \( L \) as ratios of output, e.g., \( k_X \equiv K_X / X \). It follows that

(1a) \[ \dot{k}_X - \dot{l}_X = \sigma_X (\dot{w} - \dot{r}) \quad \text{and} \quad (1b) \quad \dot{k}_Y - \dot{l}_Y = \sigma_Y (\dot{w} - \dot{r}) \]

By the envelope theorem, we know that

\[ rdk_X + wdl_X = 0 \quad \Rightarrow \quad \left( \frac{rk_X}{P_X} \right) \dot{k}_X + \left( \frac{wl_X}{P_X} \right) \dot{l}_X = 0 \quad \Rightarrow \]

(2a) \[ \theta_{XX} \dot{k}_X + \theta_{LX} \dot{l}_X = 0 ; \quad \text{also} \quad (2b) \quad \theta_{XY} \dot{k}_Y + \theta_{LY} \dot{l}_Y = 0 \]

(\( \theta \) is a cost share)

Finally, since \( L_X + L_Y = \bar{L} \Rightarrow l_X X + l_Y Y = \bar{L} \), we may totally differentiate to obtain:

(3a) \[ (\dot{l}_X + \dot{X}) \lambda_{LX} + (\dot{l}_Y + \dot{Y}) \lambda_{LY} = 0 ; \quad \text{also} \quad (3b) \quad (\dot{k}_X + \dot{X}) \lambda_{XX} + (\dot{k}_Y + \dot{Y}) \lambda_{YY} = 0 \]

where \( \lambda_{LX} = L_X / \bar{L} \) is the share of the economy’s labor that is used in sector \( X \), and the other terms are defined in the same manner.

Now, substitute (2a) into (1a) and (2b) into (1b) to get expressions for \( \dot{l}_X \) and \( \dot{l}_Y \) and (using the fact that the labor and capital cost shares \( \theta \) add to one for each sector, and that \( \lambda_{LX} + \lambda_{LY} = 1 \)) substitute these expressions into (3a) to obtain:

(4a) \[ \lambda_{LX} \dot{X} + \lambda_{LY} \dot{Y} = (\lambda_{LX} \theta_{XX} \sigma_X + \lambda_{LY} \theta_{XY} \sigma_Y ) (\dot{w} - \dot{r}) \]

Follow the same procedure to get expressions for \( \dot{k}_X \) and \( \dot{k}_Y \) to substitute into (3b) to obtain:

(4b) \[ \lambda_{XX} \dot{X} + \lambda_{XY} \dot{Y} = -(\lambda_{XX} \theta_{LX} \sigma_X + \lambda_{XY} \theta_{LY} \sigma_Y ) (\dot{w} - \dot{r}) , \]

and subtract (4b) from (4a) to obtain:

\[ \lambda^* (\dot{X} - \dot{Y}) = [a_x \sigma_X + a_y \sigma_Y ] (\dot{w} - \dot{r}) = \bar{\sigma} (\dot{w} - \dot{r}) \]

where \( a_x = \lambda_{LX} \theta_{XX} + \lambda_{XX} \theta_{LX} ; \quad a_y = \lambda_{LY} \theta_{XY} + \lambda_{XY} \theta_{LY} ; \quad \lambda^* = \lambda_{LX} - \lambda_{XX} \)