By definition of the elasticities of substitution in production, we know that

\[ \hat{K}_x - \hat{L}_x = \sigma_x (\hat{w} - \hat{r}) \quad \text{and} \quad \hat{K}_y - \hat{L}_y = \sigma_y (\hat{w} - \hat{r}) \]

For convenience, express \( K \) and \( L \) as ratios of output, e.g., \( k_x \equiv K_X/X \). It follows that

(1a) \( \hat{k}_x - \hat{L}_x = \sigma_x (\hat{w} - \hat{r}) \) \quad \text{and} \quad (1b) \( \hat{k}_y - \hat{L}_y = \sigma_y (\hat{w} - \hat{r}) \)

By the envelope theorem, we know \( d(rk_X + wl_X) = k_Xdr + l_Xdw \), so \( rkd_X + wdl_X = 0 \implies \)

(2a) \( r(Xk_X + w(l_X))\hat{k}_X + (w(l_X))\hat{I}_X = \theta_{kX}\hat{k}_X + \theta_{lx}\hat{I}_X = 0 \); also (2b) \( \theta_{kX}\hat{k}_y + \theta_{ly}\hat{I}_y = 0 \)

Finally, since \( L_X + L_Y = l_XX + l_Y = \bar{L}_X \), we may totally differentiate to obtain:

(3a) \( (\hat{I}_X + \hat{X})\lambda_{XX} + (\hat{I}_Y + \hat{Y})\lambda_{YY} = 0 \); also (3b) \( (\hat{k}_X + \hat{X})\lambda_{XX} + (\hat{K}_Y + \hat{Y})\lambda_{YY} = 0 \)

where \( \lambda_{XX} = L_X/\bar{L}_X \) is the share of the economy’s labor that is used in sector \( X \), and the other terms are defined in the same manner.

Now, substitute (2a) into (1a) and (2b) into (1b) to get expressions for \( \hat{I}_X \) and \( \hat{I}_Y \) and (using the fact that the labor and capital cost shares \( \theta \) add to one for each sector, and that \( \lambda_{XX} + \lambda_{YY} = 1 \)) substitute these expressions into (3a) to obtain:

(4a) \( \lambda_{XX}\hat{X} + \lambda_{YY}\hat{Y} = (\lambda_{XX}\theta_{kX}\sigma_X + \lambda_{YY}\theta_{kY}\sigma_Y)(\hat{w} - \hat{r}) \)

Follow the same procedure to get expressions for \( \hat{k}_X \) and \( \hat{k}_Y \) to substitute into (3b) to obtain:

(4b) \( \lambda_{XX}\hat{K}_X + \lambda_{YY}\hat{K}_Y = -(\lambda_{XX}\theta_{kX}\sigma_X + \lambda_{YY}\theta_{kY}\sigma_Y)(\hat{w} - \hat{r}) \), and subtract (4b) from (4a) to obtain:

\[
(\hat{w} - \hat{r}) = \frac{\lambda^*}{a_x\sigma_X + a_y\sigma_Y}(\hat{X} - \hat{Y}) = \frac{\lambda^*}{\sigma}(\hat{X} - \hat{Y})
\]

where \( a_x = \lambda_{XX}\theta_{kX} + \lambda_{kX}\theta_{kX} \); \( a_y = \lambda_{kY}\theta_{kY} + \lambda_{kY}\theta_{kY} \); \( \lambda^* = \lambda_{XX} - \lambda_{kX} \)