Derivation of Factor Price Frontier Expression for the Two-Sector Incidence Model

By definition of the elasticities of substitution in production, we know that
\[ \hat{K}_x - \hat{L}_x = \sigma_x (\hat{w} - \hat{r}) \quad \text{and} \quad \hat{K}_y - \hat{L}_y = \sigma_y (\hat{w} - \hat{r}) \]

For convenience, express \( K \) and \( L \) as ratios of output, e.g., \( k_x \equiv K_x/X \). It follows that

1a) \( \hat{k}_x - \hat{I}_x = \sigma_x (\hat{w} - \hat{r}) \) and 1b) \( \hat{k}_y - \hat{I}_y = \sigma_y (\hat{w} - \hat{r}) \)

By the envelope theorem, we know \( d(rk_x + wl_x) = k_x dr + l_x dw \), so \( rdk_x + wdl_x = 0 \) \( \Rightarrow \)

2a) \( \left( \frac{r_k}{P_x} \right) \hat{k}_x + \left( \frac{w_l}{P_x} \right) \hat{l}_x = \theta_{kX} \hat{k}_X + \theta_{lX} \hat{l}_X = 0 \); also 2b) \( \theta_{kY} \hat{k}_Y + \theta_{lY} \hat{l}_Y = 0 \)

Finally, since \( L_X + L_Y = L_X + L_Y = \bar{L} \), we may totally differentiate to obtain:

3a) \( (\hat{l}_x + \hat{X}) \lambda_{LX} + (\hat{I}_Y + \hat{Y}) \lambda_{LY} = 0 \); also 3b) \( (\hat{k}_x + \hat{X}) \lambda_{kX} + (\hat{k}_Y + \hat{Y}) \lambda_{kY} = 0 \)

where \( \lambda_{LX} = L_X / \bar{L} \) is the share of the economy’s labor that is used in sector \( X \), and the other terms are defined in the same manner.

Now, substitute (2a) into (1a) and (2b) into (1b) to get expressions for \( \hat{l}_x \) and \( \hat{l}_y \) and (using the fact that the labor and capital cost shares \( \theta \) add to one for each sector, and that \( \lambda_{LX} + \lambda_{LY} = 1 \)) substitute these expressions into (3a) to obtain:

4a) \( \lambda_{LX} \hat{X} + \lambda_{LY} \hat{Y} = (\lambda_{LX} \theta_{kX} \sigma_X + \lambda_{LY} \theta_{kY} \sigma_Y )(\hat{w} - \hat{r}) \)

Follow the same procedure to get expressions for \( \hat{k}_x \) and \( \hat{k}_y \) to substitute into (3b) to obtain:

4b) \( \lambda_{kX} \hat{X} + \lambda_{kY} \hat{Y} = -(\lambda_{kX} \theta_{kX} \sigma_X + \lambda_{kY} \theta_{kY} \sigma_Y )(\hat{w} - \hat{r}) \), and subtract (4b) from (4a) to obtain:

\[ (\hat{w} - \hat{r}) = \frac{\lambda^*}{a_x \sigma_X + a_y \sigma_Y} (\hat{X} - \hat{Y}) = \frac{\lambda^*}{\sigma} (\hat{X} - \hat{Y}) \]

where \( a_x = \lambda_{LX} \theta_{kX} + \lambda_{kX} \theta_{LX} \); \( a_y = \lambda_{LY} \theta_{kY} + \lambda_{kY} \theta_{LY} \); \( \lambda^* = \lambda_{LX} - \lambda_{kX} \)