1. Consider an economy with overlapping generations, each with a single agent who lives for two periods. The interest rate is fixed at $r$. Government debt is issued at the beginning of the period, and taxes, transfer payments, and government purchases occur at the end of the period. Initially, the government has no national debt outstanding, and operates a social security system that transfers 1 unit of the numeraire commodity to the older individual from the younger individual in each period.

A. Write down the government’s year-$t$ intertemporal budget constraint (GIBC) in terms of national debt, government purchases, and government net taxes (taxes less transfers), and show that the government’s existing policy satisfies the GIBC.

B. Now write down the GIBC in its alternative formulation, in terms of the initial level of debt, government purchases, and the generational accounts for all existing and future generations. Solve for the generational account for each generation, and show that this version of the GIBC is also satisfied under current government policy.

C. Suppose that, at the end of the current period, $t$ (i.e., at the beginning of period $t+1$) the government eliminates the social security system by issuing bonds to pay for the current elderly agent’s benefit. Assuming that government services the debt using equal taxes on each future elderly generation, solve for the tax needed to satisfy the GIBC. Show that all generational accounts are the same as for the original social security system in part B.

D. Now, compare the generational accounts as of date $t+1$ for the initial social security system and the post-reform system. Do the accounts differ? Is the GIBC still satisfied?

2. Consider an economy in which there are two commodities, musical composition ($X$) and home construction ($Y$). Each is produced competitively using a single factor, labor, that is in fixed supply. Producers of $X$ face the production function $X = (\alpha e^{-\beta Y})L_x$, and producers of $Y$ face the production function $Y = \gamma L_y$, where $L = L_x + L_y$ is the total labor supply. That is, each producer perceives constant returns to scale, but the productivity of labor in sector $X$ is reduced via a negative production externality by the aggregate level of production of $Y$.

A. Derive the economy’s production possibilities frontier as an expression for $X$ in terms of $Y$, and show that the production set is not convex.

B. Letting labor be numeraire, derive the cost functions for producers of each good, $c(X;Y)$ and $c(Y)$, assuming that producers of $X$ take $Y$ as given and that producers of $Y$ ignore their impact on sector $X$. Solve for the competitive prices at given values of $X$ and $Y$.

C. Now, solve for the social cost function for $X$ and $Y$, $c(X,Y)$, and the marginal costs of $X$ and $Y$ at given production levels. What Pigouvian taxes on producers of $X$ and $Y$ would cause competitive prices to equal marginal social costs, given $X$ and $Y$?
D. Suppose that consumers are identical, with preferences that satisfy the utility function 
\[ U(X,Y) = X + Y. \] How much revenue does the Pigouvian tax raise at the social optimum?

3. Consider an economy in which every family has preferences over two goods, education and housing, defined by the common utility function, 
\[ U(E,H) = E^\alpha H^{1-\alpha}. \] Households differ only with respect to income level, with household \( i \)'s exogenous income equal to \( y^i \). Housing is produced subject to constant unit cost \( p_H = 1 \), and may be purchased in any quantity. Education may be produced using one of two technologies: by the private sector as a regular private good with unit cost \( p_E = 1 \) per family, and by the public sector as a pure public good with unit cost \( q \). The level of public education, \( \bar{E} \), is financed by a proportional tax at rate \( \tau \) on income, and no individual household may use public and private education at the same time.

A. For fixed values of the tax rate, \( \tau \), and the level of public education, \( \bar{E} \), show that there exists a critical level of income, \( y^* \), above which households choose private school, and below which households choose public school. Show that \( y^* \) is increasing in \( \bar{E} \), given \( \tau \).

B. Start with your solution for \( y^* \) as a function of \( \bar{E} \) and \( \tau \) from part A. Letting \( Y \) equal aggregate income in the economy, substitute for \( \tau \) using the government’s budget constraint that relates \( \tau \) to \( \bar{E} \) and \( Y \). Calculate the full effect of \( \bar{E} \) on \( y^* \). Show that this effect is larger than the partial effect you solved for in part A, and explain why.

C. Show that, among individuals who choose public education, there is a single level of public education, say \( \bar{E}^* \), that is most preferred by all. If the existing level of public education is initially slightly below \( \bar{E}^* \), under what condition would a majority vote to increase public education spending to \( \bar{E}^* \)? (Hint: relate \( y^* \) to the income of the median voter.)

4. Suppose that a risk-neutral investor faces a tax rate of \( c \) on capital gains, while facing a tax rate of \( t \geq c \) (i.e., getting a refund at rate \( t \)) on long-term capital losses. The investor has an asset originally purchased for \( P_0 \) that is now worth \( P_1 > P_0 \), and must decide whether to (1) sell the asset now, pay a tax on \( (P_1 - P_0) \) at rate \( c \), and reinvest the remaining proceeds for one more period, or (2) continue holding the asset for one more period. In either case, the rate of return over the next period is \( r \), which is stochastic. Under choice (1), subsequent gains will be taxed at \( c \) and subsequent losses will be taxed at \( t \). Under choice (2), total gains, \( (P_1(1+r) - P_0) \) will be taxed at rate \( c \), for we assume that \( P_1(1+r) > P_0 \) even if \( r \) is negative.

A. Derive a necessary and sufficient condition for the investor to realize the capital gain now, expressed in terms of some critical value of the ratio \( R = P_1/P_0 \), say \( R^* \).

B. Assuming that there is a distribution of values of \( R = P_1/P_0 \) in the population, a lower value of \( R^* \) will mean that fewer gains will be realized immediately. Show that \( dR^*/dc < 0 \), starting from the case in which \( t \) and \( c \) are initially equal.