From now on we will write the budget constraint in "period 2 dollars". This choice is arbitrary but it ends up simplifying the algebra, because this way, its "as if" we have 2 goods with prices $(1+r)$ and 1 (rather than prices 1 and $1/(1+r)$, which would be the case in period 1 dollars).

The consumer solves \( \max U(c_1, c_2) \text{ s.t. } (1+r)c_1 + c_2 = (1+r)y_1 + y_2 \).

The Lagrangean is

\[
L(c_1, c_2, \lambda) = U(c_1, c_2) - \lambda \left( (1+r)c_1 + c_2 - (1+r)y_1 - y_2 \right)
\]

\[
\frac{\partial L}{\partial c_1} = U_1(c_1, c_2) - \lambda (1+r) = 0
\]

\[
\frac{\partial L}{\partial c_2} = U_2(c_1, c_2) - \lambda = 0
\]

\[
\frac{\partial L}{\partial \lambda} = -(1+r)c_1 - c_2 + (1+r)y_1 + y_2 = 0
\]

These give a "tangency" condition \( U_1/U_2 = (1+r) \) and the budget constraint (as usual). The solution functions in terms of \( r \) and \( (y_1, y_2) \) are:

\[
c_1 = c_1(r, y_1, y_2)
\]

\[
c_2 = c_2(r, y_1, y_2)
\]

Note that we could also define optimal choices for fixed "period 2" wealth \( W \):

\[
c_1 = c_1^w(r, W),
\]

\[
c_2 = c_2^w(r, W).
\]

These optimal choice functions are related by:

\[
c_1(r, y_1, y_2) = c_1^w(r, (1+r)y_1 + y_2)
\]

\[
c_2(r, y_1, y_2) = c_2^w(r, (1+r)y_1 + y_2)
\]

You can see there is an interesting feature of \( c_1(r, y_1, y_2) \): it is like a "fixed wealth" consumption function, with \( W = (1+r)y_1 + y_2 \) which increases with \( r \) (like "full income increases with the wage in a labor supply example).
Now let's define the expenditure function as the minimum cost to get a target utility (again, measured in period 2 dollars). Specifically, define:

\[ e(r,u^0) = \min (1+r) c_1 + c_2 \quad \text{s.t.} \quad U(c_1, c_2) = u^0. \]

The Lagrangean is:

\[ L( c_1, c_2, \mu) = (1+r) c_1 + c_2 - \mu (U(c_1, c_2) - u^0). \]

\[ \frac{\partial L}{\partial c_1} = (1+r) - \mu U_1(c_1, c_2) = 0 \]
\[ \frac{\partial L}{\partial c_2} = 1 - \mu U_2(c_1, c_2) = 0 \]
\[ \frac{\partial L}{\partial \mu} = -U(c_1, c_2) + u^0 = 0. \]

The solution functions are the compensated demands, \( c_1^c(r, u^0), c_2^c(r, u^0) \). As usual

\[ e(r,u^0) = (1+r) c_1^c(r, u^0) + c_2^c(r, u^0) \]

Differentiating,

\[ \frac{\partial e(r,u^0)}{\partial r} = c_1^c(r, u^0) + (1+r) \frac{\partial c_1^c}{\partial r} + \frac{\partial c_2^c}{\partial r} \]

and (as usual) we will be able to show** that \((1+r) \frac{\partial c_1^c}{\partial r} + \frac{\partial c_2^c}{\partial r} = 0\) so

\[ \frac{\partial e(r,u^0)}{\partial r} = c_1^c(r, u^0). \]

**remember the proof: Start with \( U(c_1^c(r, u^0), c_2^c(r, u^0)) = u^0 \). Differentiate w.r.t. \( r \), and look at the f.o.c. for the expenditure min problem.

Now we have 3 functions:

\[ c_1(r, y_1, y_2) \]
\[ c_1^w(r, W) \]
\[ c_1^c(r, u^0) \]

We also have 2 relations:

\[ c_1(r, y_1, y_2) = c_1^w(r, (1+r) y_1 + y_2) \]
\[ c_1^c(r, u^0) = c_1^w(r, e(r, u^0)) \]
Note why we had to define $c^w$: it’s the function that links the compensated demand and the demand we are ultimately interested in, $c^1(r, y_1, y_2)$.

We can differentiate these 2 functions w.r.t. $r$. Starting with the first:

$$\frac{\partial c^1(r, y_1, y_2)}{\partial r} = \frac{\partial c^w(r, (1+r)y_1 + y_2)}{\partial r} + y_1 \frac{\partial c^w(r, (1+r)y_1 + y_2)}{\partial W}$$

This shows that when you change $r$, the reaction of the demand for $c^1$ as a function of $(r, y_1, y_2)$ has an income effect, reflecting the fact that as $r$ rises, the value of wealth rises.

From the second we get an expression like we’ve seen before:

$$\frac{\partial c^1(r, u^0)}{\partial r} = \frac{\partial c^w(r, e(r, u^0))}{\partial r} + \frac{\partial c^w(r, e(r, u^0))}{\partial W} \times \frac{\partial e(r, u^0)}{\partial r}$$

Re-arranging, we get a Slutsky equation for $c^w$:

$$\frac{\partial c^w(r, e(r, u^0))}{\partial r} = \frac{\partial c^1(r, u^0)}{\partial r} - \frac{\partial c^w(r, e(r, u^0))}{\partial W} \times c^1(r, u^0).$$

assuming $u^0$ is the level of utility you can get with incomes $(y_1, y_2)$ and interest rate $r$.

Finally, we can plug this in to the expression above to get:

$$\frac{\partial c^1(r, y_1, y_2)}{\partial r} = \frac{\partial c^w(r, (1+r)y_1 + y_2)}{\partial r} + y_1 \frac{\partial c^w(r, (1+r)y_1 + y_2)}{\partial W}$$

$$= \frac{\partial c^1(r, u^0)}{\partial r} + \frac{\partial c^w(r, e(r, u^0))}{\partial W} \times (y_1 - c^1(r, y_1, y_2))$$

$$= \frac{\partial c^1(r, u^0)}{\partial r} + \frac{\partial c^w(r, e(r, u^0))}{\partial W} \times S^1(r, y_1, y_2)$$

where $S^1(r, y_1, y_2) = y_1 - c^1(r, y_1, y_2)$ is optimal first period saving.

Depending on whether $S^1$ is positive or negative, the income effect of a rise in $r$ on optimal consumption $c^1(r, y_1, y_2)$ is positive or negative.

For a “saver”, $S^1 > 0$ and the rise in $r$ has a positive income effect (because the consumer is a “net supplier” of funds to the market, as in a labor supply example).

But for a “borrower” $S^1 < 0$ and the rise in $r$ has a negative income effect (because the consumer is a “net demander” of funds, as in the basic commodity demand case.