Economics 250a
Lecture 2

Outline
0. Overview of Labor Supply Patterns
1. Static Labor Supply - basic results
2. Two applications of the expenditure function
3. Functional form - the Stern "catalogue"
4. Identification Problems
5. Addressing Non-Participation (Basic approach)

Some recommended readings:

Nicolas Stern. "On the Specification of Labour Supply Functions." In Richard Blundell and Ian Walker, editors, Unemployment Search and Labour Supply. Cambridge: Cambridge University Press, 1986.

James Heckman. "Shadow Prices, Market Wages and Labor Supply." Econometrica 42 (July 1974), pp. 679-694.

Orley Ashenfelter. "Determining Participation in Income-Tested Social programs." Journal of the American Statistical Association, Vol. 78, No. 383, September 1983.

Jerry Hausman, "Exact Consumer's Surplus and Deadweight Loss" Am Econ Review 71 (Sept 1981), 662-676

George Borjas "The Relationship between Wages and Weekly Hours of Work: The Role of Division Bias". Journal of Human Resources, 15 (3), 1980 409-423.

In addition, I recommend the Handbook of Labor Economics chapters by Pencavel (volume 1) and Blundell and MaCurdy (volume 3a), as well as the more recent chapter:

Richard Blundell, Thomas MaCurdy and Costas Meghir. "Labor Supply Models: Unobserved Heterogeneity, Nonparticipation, and Dymanics" Handbook of Econometrics volume 6, 2007.

0. Overview of Data

In problem set 2 you will get a chance to explore some basic data on annual labor supply patterns from the Current Population Survey (CPS). In the meantime, the figures at the end of this lecture show average patterns of work activity by age for men and women. Between the ages of 25 and 55 a large share of men (90%) and a majority of women (75-80%) work at least a little. A typical male in that age range (in a "good" year) works ~2000 hours (or at least reports to work about 40 hours a week, 50 weeks a year). Women are more diverse, with some working about 40 hours a week, others working about 20, so the average is about 1400-1500 per year. One important goal of static labor supply modeling is to explain the variation in hours choices, and how choices are affected by public policies like welfare benefits, taxes, etc. A static framework is arguably most relevant for people whose current situation is representative of their "permanent" situation: it is also appropriate if people have no access to borrowing/lending. Lifecycle labor supply models, which we will discuss in subsequent lectures, attempt to explain multiple phenomenon, including: (i) the pattern of the age profile of hours (ii) the decision of when to stop working altogether (retirement) (iii) year-to-year variation in hours choices.

1. Static Labor Supply - Basic Results

The basic setup for a static labor supply problem has a single agent who will definitely work at least some hours (i.e., an "interior solution"). The agent has a strictly increasing, strictly quasi-concave utility function $u(x, \ell)$ where $x =$ consumption of goods and services (sold at price p, which will be set to 1 in many contexts) and $\ell =$ "leisure".** We assume $\ell \in [0, T]$, and $x \geq 0$. Hours of work are $h = T - \ell$,the agent has "non-labor income" y, and faces a parametric hourly wage is w. The agent's budget constraint is:

$$\begin{aligned} px &= wh + y \\ &= w(T - \ell) + y \qquad \text{or re-arranging,} \\ px + w\ell &= wT + y. \end{aligned}$$

Sometimes the amount $wT + y$ is referred to as "full income" since this is the amount of money available to purchase either goods or leisure. Importantly, full income depends on w, generating an income effect in response to a rise in w that is the opposite sign to the standard case.

**If you want to write utility directly in terms of hours, think of specifying $v(x; h) = u(x; T - h)$: Since there is a 1:1 linear relation, the choice is arbitrary.

The direct approach to finding the agent's labor supply function is to maximize utility subject to the budget constraint. The maximized value function (i.e., the indirect utility function) is

$$v(p, w, y) = \max_{x,h} u(x, T - h) \;\; \text{s.t.} \quad px = wh + y$$

Set up the Lagrangean expression:

$$L(x, h, \lambda; p, w, y) = u(x, T - h) - \lambda(px - wh - y)$$

Assuming an interior optimum the first order conditions are

$$\begin{aligned} L_x &= u_x(x, T - h) - \lambda p = 0 \\ L_h &= -u_\ell(x, T - h) + \lambda w = 0 \\ L_h &= -px + wh + y = 0 \end{aligned}$$

The first two conditions can be rewritten as

$$\frac{u_\ell(x, T - h)}{u_x(x, T - h)} = \frac{w}{p}$$

which expresses the 'tangency' condition that the marginal rate of substitution ($mrs = u_\ell/u_x$) equals the real wage. The solution functions to the direct optimization problem are:

$$\begin{gathered} h(p, w, y) \\ x(p, w, y) \\ \lambda(p, w, y) \end{gathered}$$

with $\ell(p, w, y) = T - h(p, w, y)$ defining the demand for leisure. We refer to the function $h(p, w, y)$ as the labor supply function of the agent. As in the basic consumer demand case, $h(p, w, y)$ is HD_0 in (p, w, y). Also, from the envelop theorem, $v_w(p, w, y) = \lambda h$, $v_y(p, w, y) = \lambda$, so $h(p, w, y) = v_w(p, w, y)/v_y(p, w, y)$, which is Roy's identity for labor supply.

The indirect approach is to define the expenditure function (sometimes called the excess exenditure function)

$$e(p,w,u) = \min px - wh \quad \text{s.t.} \quad u(x, T-h) \geq u.$$

This is amount of money you need to get *in addition to your earnings* to achieve utility u, given (p,w). The solution functions for the minimization problem are

$$h^c(p,w,u)$$
$$x^c(p,w,u)$$

where the subscript "c" denotes "compensated" (or Hicksian) demand/supply. Sheppard's lemma (i.e., the envelop theorem) states that

$$h^c(p,w,u) = -\frac{\partial e(p,w,u)}{\partial w}.$$

Note the sign change from the usual case: a rise in the wage causes e to fall: the derivative is just $-h^c$. Since this equation holds as we vary (p,w), we can differentiate to get

$$\frac{\partial h^c(p,w,u)}{\partial w} = -\frac{\partial^2 e(p,w,u)}{\partial w^2} > 0$$

($= 0$ for the Leontief case, which is not covered by our assumptions on u).

From an initial set of conditions (p^0, w^0, y^0) let $u^0 = v(p^0, w^0, y^0)$, and let $h^0 = h(p^0, w^0, y^0)$. Then locally we have

$$h^c(p^0, w, u^0) = h(p^0, w, e(p^0, w, u^0)).$$

Differentiating with respect to w:

$$\frac{\partial h^c(p^0, w^0, u^0)}{\partial w} = \frac{\partial h(p^0, w^0, y^0)}{\partial w} + \frac{\partial h(p^0, w^0, y^0)}{\partial y}\frac{\partial e(p^0, w^0, u^o)}{\partial w}$$

and since

$$\frac{\partial e(p^0, w^0, u^o)}{\partial w} = -h^c(p^0, w^0, u^o) = -h(p^0, w^0, y^o)$$

we get the Slutsky equation

$$\frac{\partial h(p^0, w^0, y^0)}{\partial w} = \frac{\partial h^c(p^0, w^0, u^0)}{\partial w} + \frac{\partial h(p^0, w^0, y^0)}{\partial y} h(p^0, w^0, y^o)$$

or in elasticity form

$$\frac{w^0}{h^0}\frac{\partial h(p^0, w^0, y^0)}{\partial w} = \frac{w^0}{h^0}\frac{\partial h^c(p^0, w^0, u^0)}{\partial w} + w^0\frac{\partial h(p^0, w^0, y^0)}{\partial y}$$

$$\epsilon = \epsilon^c + w^0 \frac{\partial h}{\partial y}$$

Now from the budget constraint

$$\begin{aligned} px &= wh + y \\ p\frac{\partial x}{\partial y} &= w\frac{\partial h}{\partial y} + 1 \\ w\frac{\partial h}{\partial y} &= -(1 - p\frac{\partial x}{\partial y}) \\ &= -(1 - mpe) \end{aligned}$$

where mpe is the marginal increase in total spending on goods and services if non-labor income goes up by \$1. Note that $(1-mpe)$ is the marginal increase in total spending on leisure when non-labor income rises by \$1. Since leisure is assumed to be a normal good, $(1-mpe)>0$, although it may be a relatively small number. You can also interpret $w\frac{\partial h}{\partial y}$ as the change in total earnings if you receive \$1 of additional non-labor income. To translate it into a more interpretable number, imagine you were to receive \$1 per year extra income for the rest of your life. How much would you reduce your average earnings per year?

The classic benchmarks (for male workers in US) are $\epsilon^c \approx 0.10-0.30$ and $(1-mpe)\approx 0.1-0.2$ implying $\epsilon \in [-0.10, 0.20]$.

c. Derivation of Slutsky Using the Utility Function

Often we are interested in understanding how a particular specification of the utility function maps to behavioral labor supply responses. In this section we relate the responses to the derivatives of U. Consider the problem of maximizing $U(x;h)$ s.t. $x=wh+y$. Note that I have set $p=1$. The f.o.c. from the Lagranean are:

$$\begin{aligned} U_1(x;h)-\lambda &= 0 \\ U_2(x;h)+\lambda w &= 0 \\ -x+wh+y &= 0. \end{aligned}$$

Differentiating we get:

$$\begin{bmatrix} U_{11} & U_{12} & -1 \\ U_{21} & U_{22} & w \\ -1 & w & 0 \end{bmatrix} \begin{bmatrix} dx \\ dh \\ d\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \\ -h & -1 \end{bmatrix} (dw, dy).$$

The determinant of the l.h.s. matrix is

$$\Delta = -(U_{22}+2wU_{12}+w^2U_{11}).$$

This has to be positive from the s.o.c. that the second-order effect of a budget neutral change (dx, dh) has to be strictly negative. To see his, notice that a budget-neutral variation has $dx = wdh$, so we have $(dx, dh) = (w, 1)dh$. The second-order effect of such a variation is

$$(w,1)\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix} = w^2U_{11}+2wU_{12}+U_{11}<0.$$

Using Cramer's rule and the fact that $\lambda = U_1$:

$$\frac{\partial h}{\partial w} = \Delta^{-1}\det \begin{bmatrix} U_{11} & 0 & -1 \\ U_{21} & -U_1 & w \\ -1 & -h & 0 \end{bmatrix} = \frac{U_1+h(wU_{11}+U_{21})}{\Delta}$$

$$\frac{\partial h}{\partial y} = \Delta^{-1}\det \begin{bmatrix} U_{11} & 0 & -1 \\ U_{21} & 0 & w \\ -1 & -1 & 0 \end{bmatrix} = \frac{wU_{11}+U_{21}}{\Delta},$$

so from the Slutsky equation we can infer that:

$$\frac{\partial h^c}{\partial w} = \frac{\partial h}{\partial w} - h\frac{\partial h}{\partial y} = \frac{U_1}{\Delta} = \frac{U_1}{-[U_{22}+2wU_{12}+w^2U_{11}]} \geq 0$$

or in terms of U only, using the fact that $w = -U_2/U_1$, we have:

$$\frac{\partial h^c}{\partial w} = \frac{-(U_1)^3}{(U_1)^2U_{22} - 2U_1U_2U_{12} + (U_2)^2U_{11}}(*).$$

A utility function that is s.q.c. will satisfy $\frac{\partial h^c}{\partial w} > 0$. (You can check this is one term of the "bordered Hessian test" for sqc).

2. Two applications of the expenditure function

a. Deadweight loss of taxation

From initial conditions $(p^0 = 1, w^0, y^0)$, suppose a proportional tax t is introduced. Let $h' = h(w^0(1-t), y^0)$, and let $u' = v(w^0(1-t), y^0)$. Suppose that the before tax wage w^0 is unaffected (no G.E. effects). The equivalent variation for the change is

$$EV = e(w^0(1-t), u') - e(w^0, u')$$

Take a second order expansion:

$$\begin{aligned} e(w^0, u') &\approx e(w^0(1-t), u') + tw^0 \frac{\partial e(w^0(1-t), u')}{\partial w} + .5t^2(w^0)^2 \frac{\partial^2 e(w^0(1-t), u')}{\partial w^2} \\ &= e(w^0(1-t), u') - tw^0h' - .5t^2w^0h' \frac{w^0}{w^0(1-t)} \left(\frac{w^0(1-t)}{h'} \frac{\partial h^c(w^0(1-t), u')}{\partial w}\right) \end{aligned}$$

So

$$\begin{aligned} EV &\approx tw^0h' + .5t^2w^0h' \frac{w^0}{w^0(1-t)} \epsilon^c \\ &\approx tw^0h'(1 + .5t\epsilon^c) \end{aligned}$$

The equivalent variation exceeds the amount of taxes collected by roughly $\frac{100}{2} \times t\epsilon^c$ percent.

b. Opting-in to welfare

A stylized welfare program has 2 parameters: G = the minimum or guaranteed level of income, and t, the "clawback rate" of benefits. So an individual with earnings E receives benefit = $\max[0, G - tE]$. Note that there is a 'breakeven' level of earnings $B = G/t$ such that if $E \geq B$, the individual is out of the welfare system. In the absence of welfare a certain individual faces a wage w^0, has non-labor income of 0, works h^0 hours, and has utility $u^0 = v(w^0, 0)$. Clearly, anyone with $E = w^0h^0$ in the absence of welfare should participate (and in fact may actually reduce hours).

But if $\epsilon^c > 0$, some people who would earn $E > B$ in the absence of the program will reduce their hours to "opt in". What is the predicted cutoff? Ashenfelter (1983) noted an agent will opt in if $e(w^0(1-t), u^0) \leq G$. This is illustrated in Figure 2.1.

Taking a second order expansion around $e(w^0, u^0) = 0$, we get:

$$e(w^0(1-t), u^0) = e(w^0, u^0) + tw^0h^0 + .5t^2w^0h^0\epsilon^c$$

Thus an agent will participate iff

$$\begin{aligned} tw^0h^0 - .5t^2w^0h^0\epsilon^c &\leq G \text{ or} \\ w^0h^0 &\leq \frac{G/t}{1 - .5t\epsilon^c} \end{aligned}$$

Note that when $\epsilon^c = 0$ (the Leontief case) this is the cutoff $w^0 h^0 \leq B = G/t$. If $\epsilon^c = 0.2$ and $t = .5$ (for example) the cutoff is (approximately) $1.05B$: people with initial earnings within 5% of the breakeven are willing to cut their hours to opt in. (The marginal entrant will reduce her hours by approximately $100 \times t\epsilon^c$ percent). Ashenfelter analyzes the Seattle-Denver NIT experiment, where the treatment group was allowed to access a welfare program similar to this simple 2-parameter system. He uses the experimental data to estimate opt-in behavior and obtains an estimate of ϵ^c in the "benchmark" range.

3. Functional Form: Stern's catalogue.

Stern (1986) presents a very useful catalogue of functional forms for static single-agent labor supply modeling. For a variety of ad hoc functional forms for the labor supply function (e.g., linear, log-linear, partially log-linear) he uses the integrability theorem to actually figure out the associated expenditure and utility functions. He also shows the labor supply functions for some leading examples of utility functions (LES, quadratic, CES) and for the translog indirect utility function. Finally, he briefly discusses some of the pluses and minuses of the different functional forms, including conditions for Slutsky terms to be consistent with theory. I recommend you read this article and keep it in your files for future reference, in case you ever have to think about a functional form to choose.

As an example, Stern (Table 9.6) considers the following very useful "semi-log" functional form

$$h = \alpha \log w + \beta y + \gamma$$

He shows that the associated expenditure function has the form

$$\begin{aligned} e(w,u) &= e^{-\beta w} u - \frac{\gamma}{\beta} - \frac{\alpha}{\beta} \log w + \frac{\alpha}{\beta} e^{-\beta w} \operatorname{Ei}(\beta w), \text{ where} \\ \operatorname{Ei}(x) &\equiv \int_{-\infty}^{x} \frac{e^t}{t} dt \text{ is the so-called exponential integral function.} \end{aligned}$$

This functional form is nice because (i) it easily handles zero or negative values for y, (ii) as we will see later, it can be used in a Tobit formulation to handle non-participation. With this functional form it is natural to introduce heterogeneity in the γ term, e.g., $\gamma = Z\theta + \eta$. (note that with unbounded support for η some people are predicted not to work - this can be an advantage or a disadvantage).

Stern (Table 9.3) also discusses the linear supply function which was "integrated" by Hausman (AER, 1981) and has been used for various applications in the literature.

Exercise : Work through Table 9.9 in Stern, for the LES.

You may also want to look at Table 9.14, where Stern discusses a LES-style "household production" model in which people get utility from "latent goods" (sometimes called "z-goods") that are a combination of time inputs and purchased inputs. (This is adapted from A. Atkinson and N. Stern, "A Note on the Allocation of Time", Economics Letters, 3 (1979): 119-123).

4. Identification Problems

A key problem in the labor supply literature is identification. Suppose for sake of discussion we assume

$$h = \alpha \log w + \beta y + Z\theta + \eta$$

where Z is a set of observed covariates and η is a "taste shock".

Problem #1: measurement error in h and w.

In many data sets we don't see the hourly wage, w. Instead we see earnings (E) for some period (last year) and total hours worked (h). Both are measured with error, but (as we will see in the exercises) hours look especially noisy. Suppose the true values for a person are (E, h), and the observed values are

$$\begin{aligned} E^o &= E + e_1 \\ h^o &= h + e_2. \end{aligned}$$

You construct a "wage" $w^o = E^o/h^o$. Note that when e_2 is positive observed hours are higher than they actually were and the observed wage is lower. Likewise when e_2 is negative, observed hours are lower than they actually were and the wage is higher. This is called the problem of "division bias": there is a mechanical negative correlation between hours and the hourly wage caused by measurement error in hours. See Borjas (1980).

Problem #2: the correlation of tastes for hours and wages.

Note that η is an unobserved component in the tastes for work. For consistent estimation of the key parameters (α, β) we need that $Cov[\log w, \eta] = 0$. In many settings this is not likely to be true. For example, someone who is unhealthy may have low hours and earn low wages. It may also be the case that $\log w$ depends on past accumulated experience. If tastes are correlated over time, people who prefer to work a lot will end up with higher wages.

Problem #3: the problem of isolating an exogenous component of y

In a typical data set we observe an individual's earnings and their non-labor income. Some important components of non-labor income are mechanically related to earnings. For example, means-tested benefits are negatively related to earnings (and therefore to hours). Other components of non-labor income are also likely to be correlated with unobserved tastes for work. For example, asset income represents a return to past savings. Many models would suggest that people with stronger tastes for work will end up working more hours each year, and with higher asset income at later stages of their life. In studies of female labor supply it is common to take husband earnings as exogenous, and use this to identify the income effect on hours. Sometimes this is also done in studies of male earnings. It is unclear whether other family members' labor incomes are uncorrelated with an individual's unobserved tastes for work.

5. Non-Participation

Return to the canonical model:

$$\max_{x,h} u(x, T - h) \quad \text{s.t.} \quad px = wh + y$$

It may happen that the max is achieved with $(h = 0, x = y)$: an endpoint optimum. This will happen if

$$mrs(y, T) \equiv \frac{u_\ell(y, T)}{u_x(y, T)} > w.$$

We can define

$$m(h;w,y) \equiv wu_x(y+wh, T-h) - u_\ell(y+wh, T-h)$$

which is the marginal value of an increase in hours at some level h, given (w, y). If $u(x, \ell)$ is s.q.c., the function $m(h; w, y)$ is strictly decreasing in h. (Exercise: prove it). The question of whether to participate is the question of whether $m(0; w, y) > 0$. For a given functional form for $u(\ )$ we can think of labor supply as generated by a "Tobit-like" model based on $m(h; w, y)$. We have:

$$\begin{aligned} m(0;w,y) &\leq 0 \Rightarrow h = 0, \\ \textit{else } h &> 0 \text{ implicitly defined by } m(h;w,y) = 0 \end{aligned}$$

One way to handle non-participation is to pick a convenient functional form for u such that this is a workable model. The classic assumption is that u is quadratic in (x, ℓ): in this case u_x and u_ℓ are both linear in h. A problem (nicely addressed by Heckman, 1974, which we discuss next) is that we don't observe w for those who don't work. So in applications, people usually end up "imputing" a wage for non-workers – not always an attractive choice.

Aside: Arthur Goldberger wrote a monograph in 1967 – Functional Form and Utility: A Review of Consumer Demand Theory (ultimately published in 1987 by Westview Press) – that works through the quadratic utility case in a lot of detail, including a nice discussion of endpoint optimums. See also Michael Ransom "An Empirical Model of Discrete and Continuous Labor Supply, Review of Economics and Statistics 69 (August 1987): 465-472 for an application to the case of family labor supply.

Heckman (1974) defines the reservation wage w^* by

$$w^* = \frac{u_\ell(y,T)}{u_x(y,T)}.$$

This is the wage that is just high enough to induce the agent to supply a tiny unit of labor. In terms of the above notation, w^* is the wage such that $m(0; w^*, y) = 0$. Note that w^* is a function of y and (in Heckman's application to married women also depends on the number and age of children). Associated with u is the labor supply function $h(w, y)$. A model for the data is

$$\begin{aligned} w &\leq w^* \Rightarrow h = 0 \\ w &> w^* \Rightarrow h = h(w,y) \end{aligned}$$

Rather than parameterizing preferences, Heckman writes down a statistical model for w_i^*, and another statistical model for wages w_i of agent i:

$$\begin{aligned} w_i &= X_i b + e_{1i} \\ w_i^* &= X_i \beta + e_{2i}. \end{aligned}$$

where X includes y and other variables. Define $z_i = w_i - w_i^* = X_i(b - \beta) + e_{1i} - e_{2i}$, and assume that the labor supply function is

$$h_i = X_i c + w_i d + v_i.$$

The likelihood for the observed data then consists of two parts:

$$\textit{Likelihood} = P(z_i \leq 0) + f(w_i, h_i | z_i > 0) \cdot P(z_i > 0)$$

where $f(w_i, h_i|z_i > 0)$ is the joint density of wages and hours conditional on $z_i > 0$ (which depends on the joint distribution of (e_{1i}, e_{2i}, v_i).

Note that if the labor supply function is $h_i = X_i c + w_i d + v_i$, then the reservation wage is the solution to

$$\begin{aligned} 0 &= X_i c + w^* d + v_i, \quad \text{or} \\ w^* &= -X_i \frac{c}{d} - \frac{1}{d} v_i \end{aligned}$$

which means that there are really only 2 latent random variables, e_{1i} and v_i, rather than three (i.e., $e_{2i} = -\frac{1}{d} v_i$).

An issue in modeling non-participation is that people are (almost) never observed working <5 hrs. per week. Either there are "fixed costs" of working that have to be overcome, or employers require some minimum hours committment to aid in co-ordination of hours schedules. We will come back to the issue of non-participation in Lecture 3, where we discuss estimation of "discretized" budget sets. A discretized framework is also very useful for thinking about part-time versus full time jobs, and for dealing with budget sets that have a lot of kinks.

6. Compensating Wage Differentials for Fixed Hours Packages

Our final application considers a classic idea (articulated by Adam Smith in *The Wealth of Nations*) that if there is a job that has limited hours (e.g., a weather-dependent job) people who take that job over another unconstrained job will have to be paid a compensating wage differential. We define

$$R(\overline{h}, u) = \min_x x \quad \text{s.t.} \quad u(x, T - \overline{h}) \geq u.$$

This is the minimum amount of consumption that in combination with $\overline{h}$ achieves utility u. R is just the vertical distance from the x-axis to the u indifference curve when $\ell = T - \overline{h}$. If a job pays the wage w and requires $\overline{h}$ hours of work then an individual would have to receive

$$R(\overline{h}, u) - w\overline{h} = \min_x x - w\overline{h} \quad \text{s.t.} \quad u(x, T - \overline{h}) \geq u$$

in additional nonlabor income to achieve utility u. Note that if $\overline{h} = h^c(w, u^0)$ then the required non-labor income is $e(w, u^0)$:

$$R(h^c(w, u^0), u^0) - wh^c(w, u^0) = e(w, u^0) \qquad (*)$$

This holds as we vary w so differentiating:

$$R_1 \frac{\partial h^c}{\partial w} - h^c - w \frac{\partial h^c}{\partial w} = \frac{\partial e}{\partial w}$$

But since $\partial e / \partial w = -h^c$, we have that

$$R_1(h^c(w, u^0), u^0) = w.$$

If you think of R as the height of the indifference curve, and recall that w is the slope of the indifference curve at $h = h^c(w, u^0)$ this is obvious. Now this relation also holds as we vary w so differentiating again

$$\begin{aligned} R_{11} \frac{\partial h^c}{\partial w} &= 1 \\ &\Rightarrow R_{11}(h^c(w, u^0), u^0) = [\frac{\partial h^c(w, u^0)}{\partial w}]^{-1} \end{aligned}$$

This shows that the inverse of the slope of the compensated labor supply curve is the rate of change of the slope of the indifference curve.

Now suppose there is an unconstrained job that pays a wage w^0, and another constrained job that requires $h = \overline{h}$. We ask: what wage w would the constrained job have to pay so an agent is indifferent between the two jobs. The difference $(w - w^0)$ is the compensating differential for the constrained choice. Using the R function we must have

$$R(\overline{h}, u^0) - w\overline{h} = e(w^0, u^0) \qquad (**)$$

Figure 2.2 shows the determination of $R(\overline{h}, u^0)$.

Now we use our standard trick - a second order expansion. In this case we will expand around $R(h^c(w^0, u^0), u^0)$, where u^0 is the utility level of the reference job. Let h^0 be the (unconstrained) hours choice on that job. We have

$$\begin{aligned} R(\overline{h}, u^0) &\approx R(h^c(w^0, u^0), u^0) + (\overline{h} - h^0)R_1(h^c(w^0, u^0), u^0) + .5(\overline{h} - h^0)^2 R_{11}(h^c(w^0, u^0), u^0) \\ &= e(w^0, u^0) + w^0 h^0 + (\overline{h} - h^0)w^0 + .5(\overline{h} - h^0)^2 [\frac{\partial h^c(w^0, u^0)}{\partial w}]^{-1} \qquad \text{(using (*) above)} \\ &= e(w^0, u^0) + \overline{h}w^0 + .5(\overline{h} - h^0)^2 [\frac{w^0 \partial h^c(w^0, u^0)}{h^0 \partial w}]^{-1} \frac{w^0}{h^0}. \end{aligned}$$

Now subtract $w\overline{h}$ from both sides:

$$R(\overline{h}, u^0) - w\overline{h} = e(w^0, u^0) - \overline{h}(w - w^0) + .5\frac{w^0}{h^0}(\overline{h} - h^0)^2 \frac{1}{\epsilon^c}$$

And using (**) we get

$$\frac{(w - w^0)}{w^0} = .5\frac{(\overline{h} - h^0)^2}{h^0 \overline{h}} \frac{1}{\epsilon^c}$$

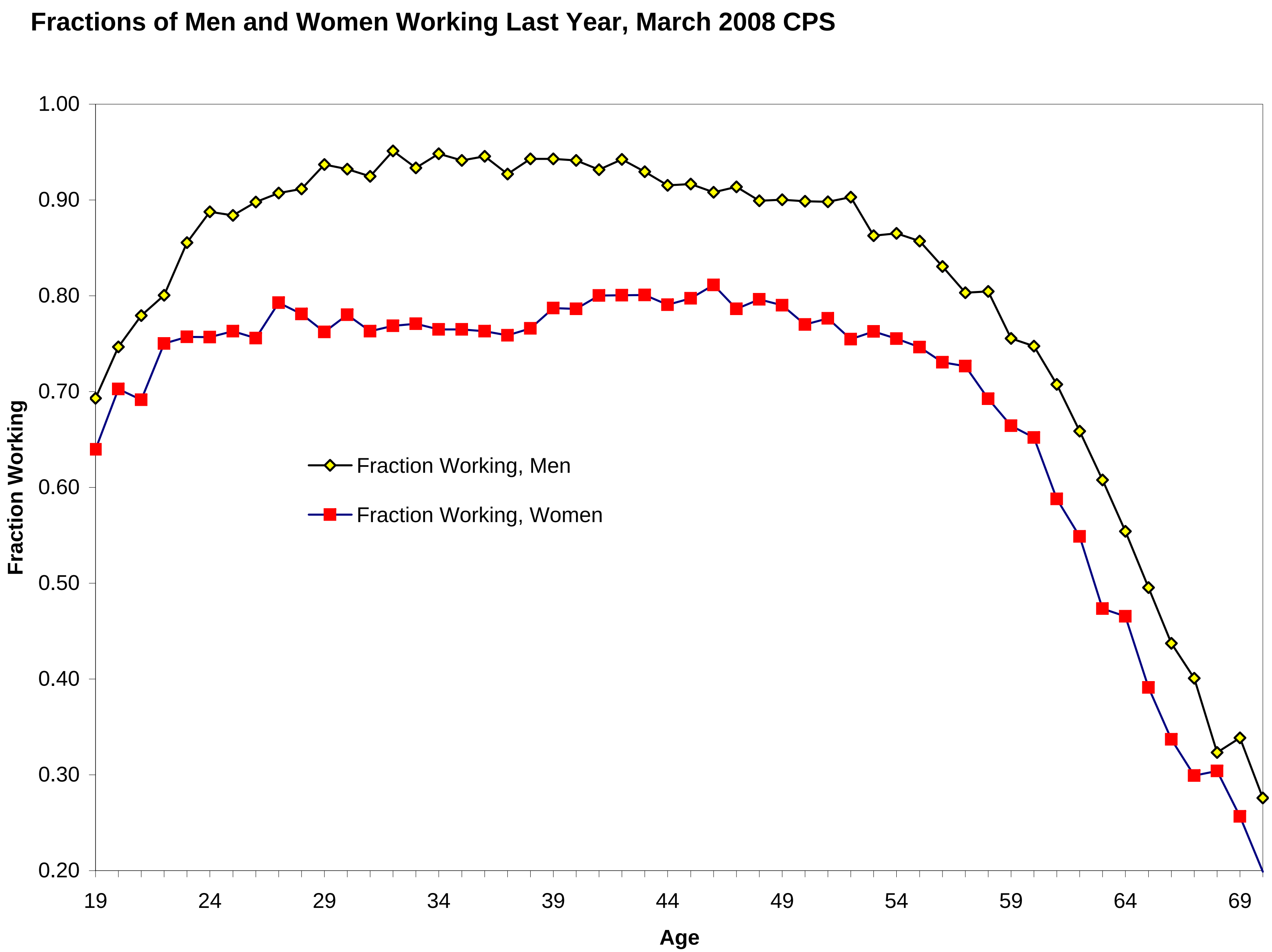
For example, if

$$\frac{(\overline{h} - h^0)}{h^0} \approx .2$$

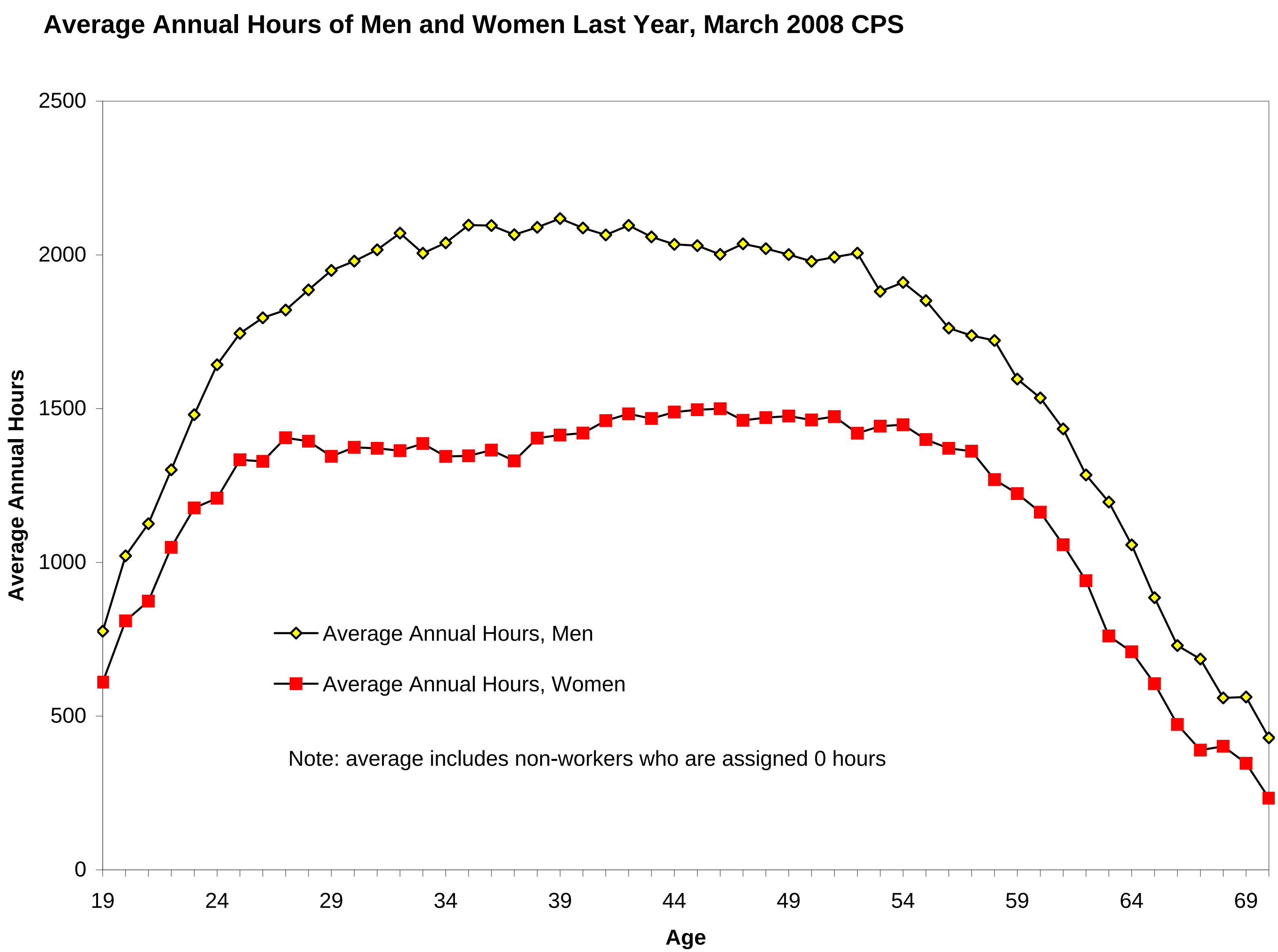
and $\epsilon^c = .2$ then the compensating differential is

$$\frac{(w - w^0)}{w^0} = \frac{.5 \times .2 \times .2}{.2} = .1$$

You need a 10% higher wage to take a job with 20% lower hours. Note that the formula also applies for high-hours jobs – in fact the formula is symmetric, so you need a 10% higher wage for job that forces you to work 20% more than you'd like. This may be a good way to introspect about your own value of ϵ^c.

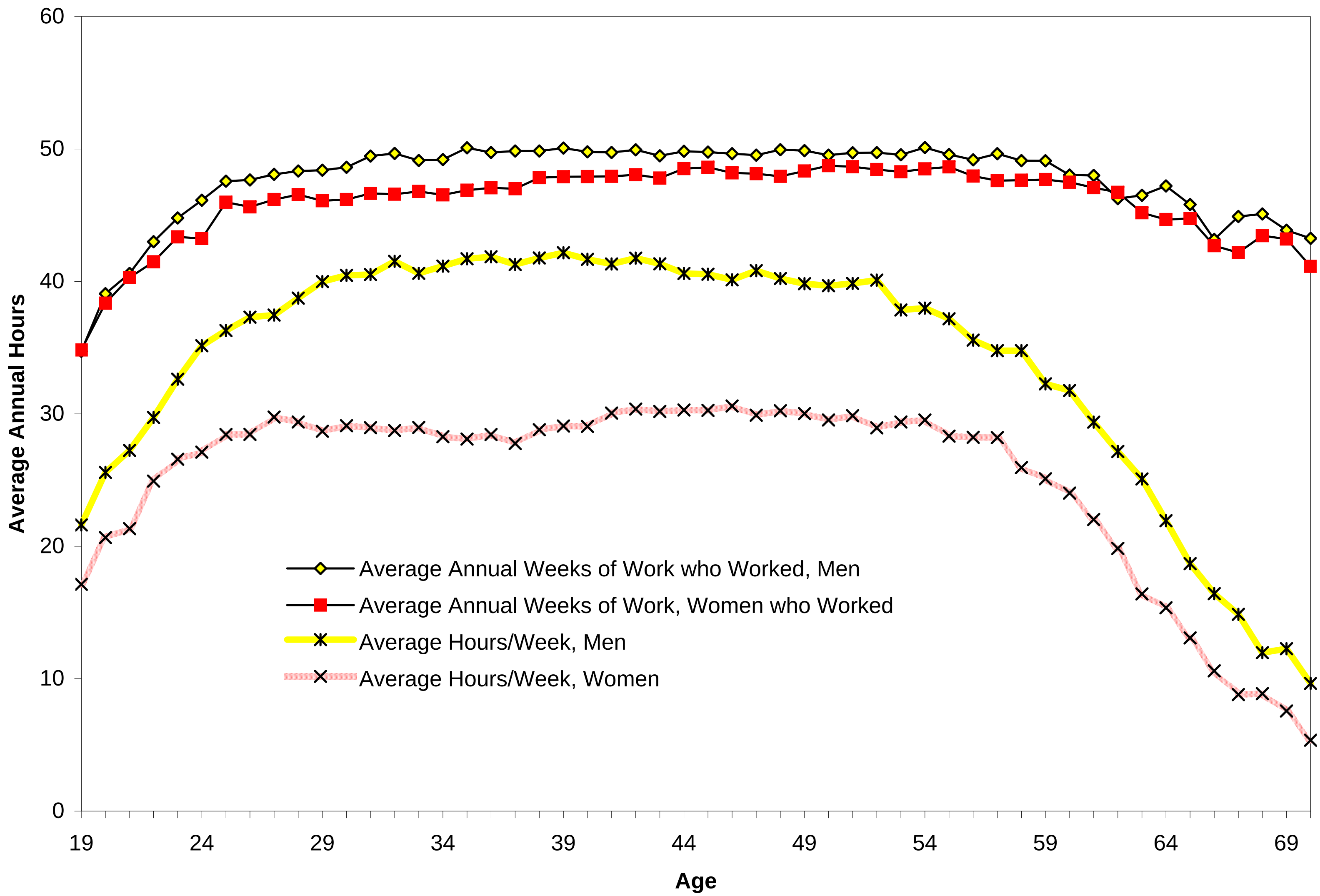


Fractions of Men and Women Working Last Year, March 2008 CPS
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
Fraction Working
Fraction Working, Men
Fraction Working, Women
19
24
29
34
39
44
49
54
59
64
69
Age



Average Annual Hours of Men and Women Last Year, March 2008 CPS
Average Annual Hours
2500
2000
1500
1000
500
0
19
24
29
34
39
44
49
54
59
64
69
Age
Average Annual Hours, Men
Average Annual Hours, Women
Note: average includes non-workers who are assigned 0 hours

Average Annual Weeks and Hours Per Week of Men and Women, March 2008 CPS



Average Annual Earnings of Men and Women Last Year, March 2008 CPS

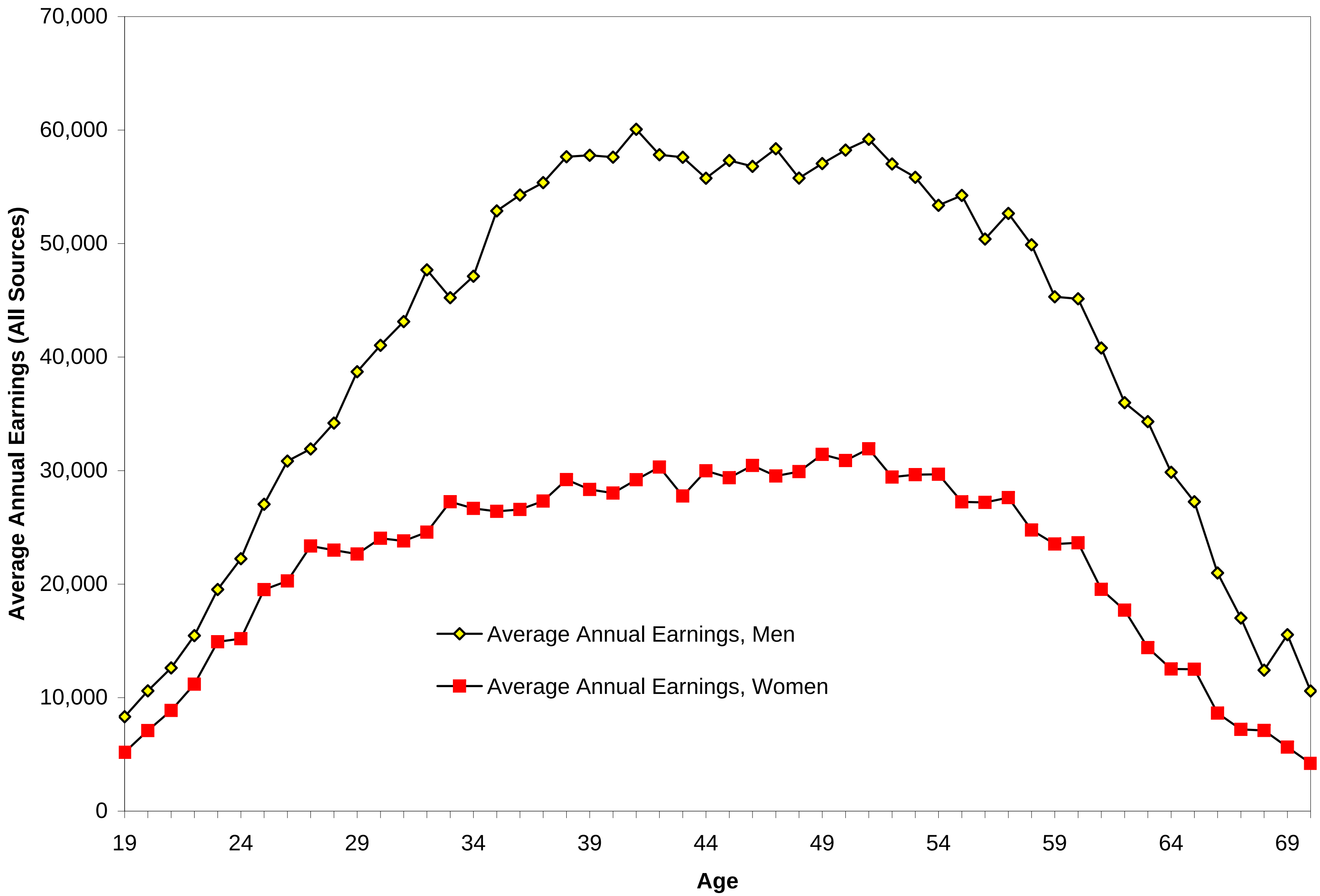


Figure 2.1: Opt-in to Welfare

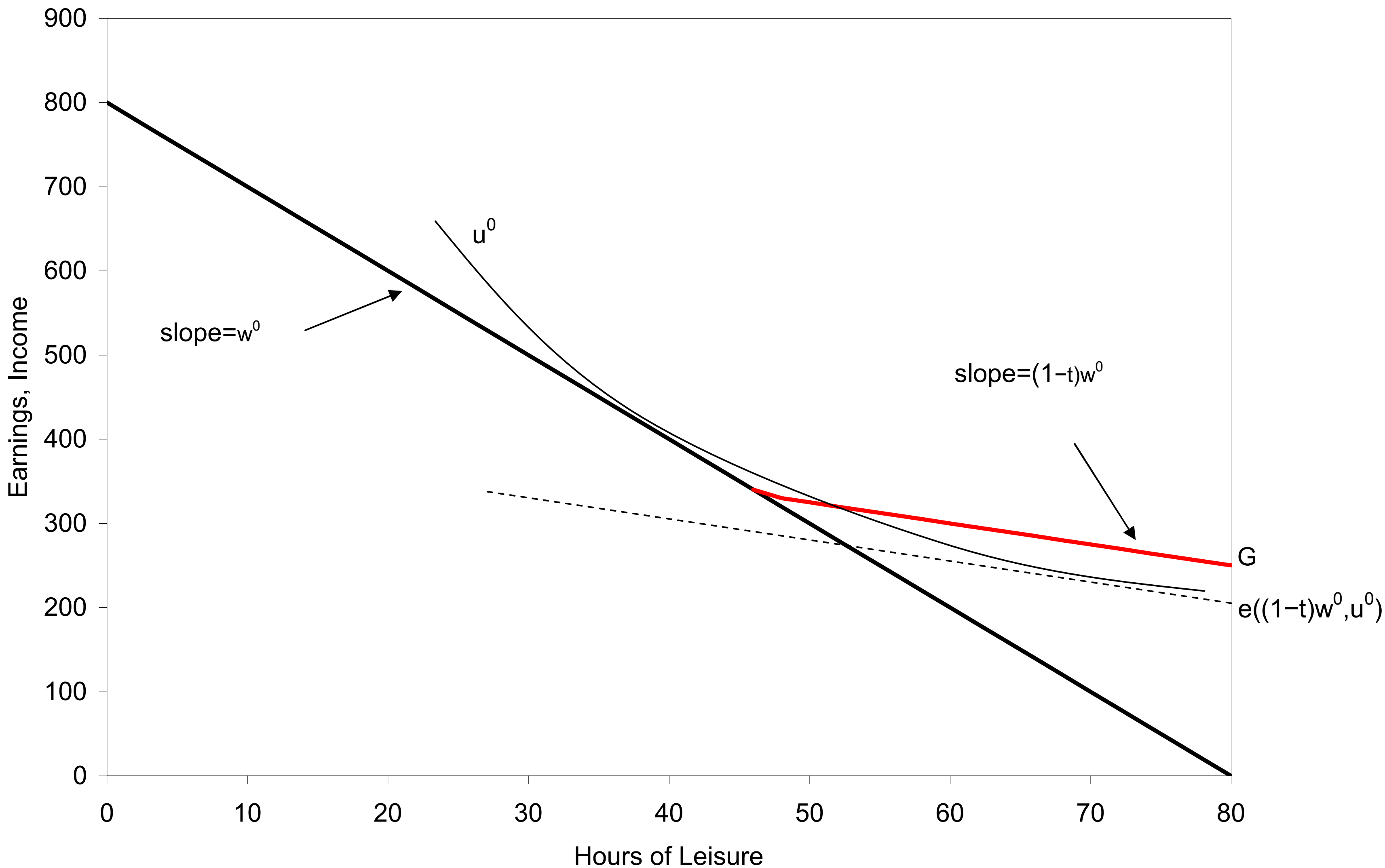
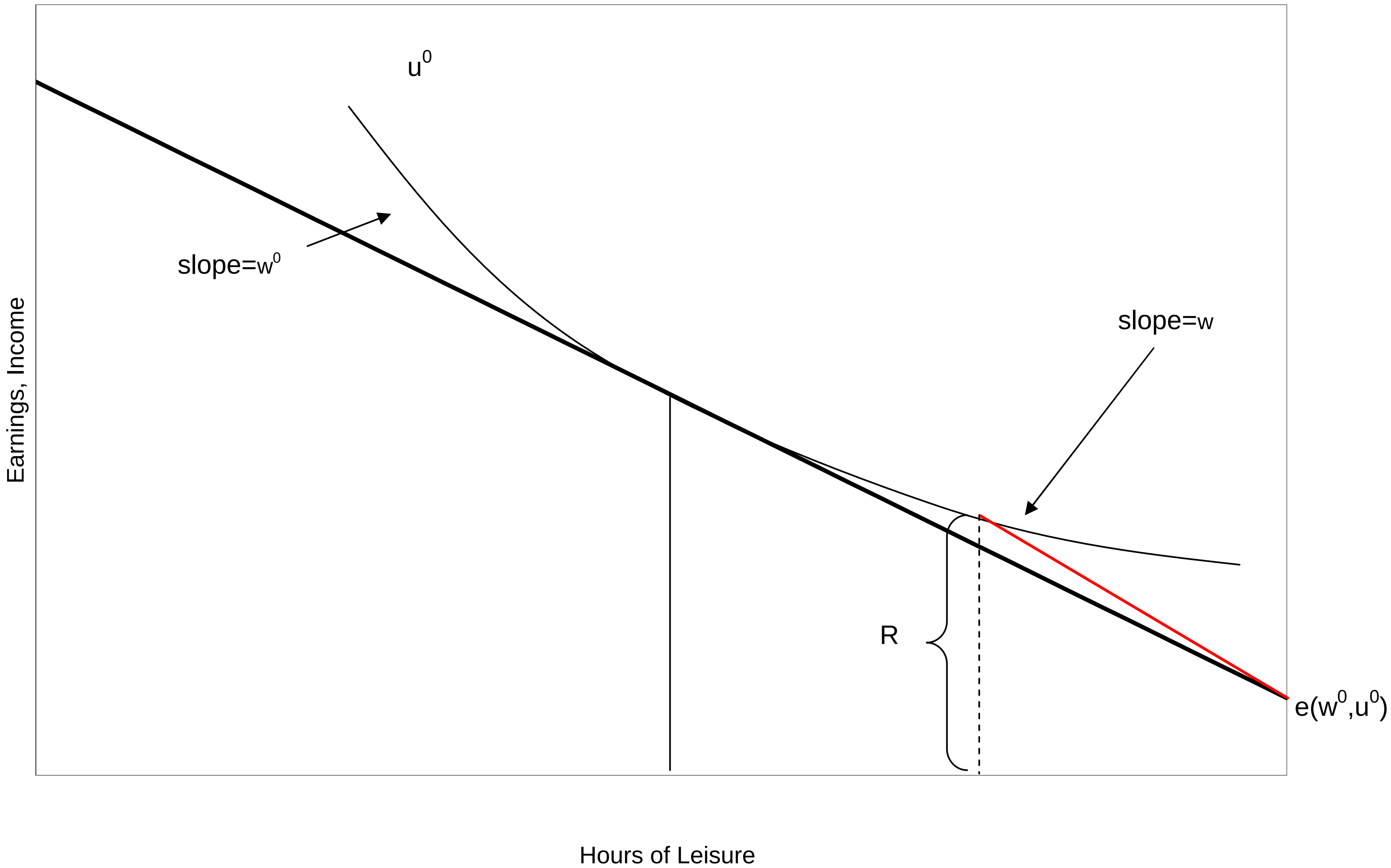


Figure 2.2: Compensating Wage Differential for Constrained Hours Choice



Economics 250a

Hausman's (1981) linear demand system – an application of integrability.

Reference: Jerry Hausman, "Exact Consumer's Surplus and Deadweight Loss" AER 71 (Sept 1981), 662-676.

Hausman considers a linear demand model for a consumer who can choose between 2-goods:

$$x = x(p, y) = \alpha p + \delta y + \gamma + \varepsilon \tag{1}$$

where x is the demand for the commodity of interest, p is the price of the commodity, y represents income, γ is a constant (that can potentially vary across people depending on demographics) and ε is a pure error term (not part of preferences; possibly measurement error in x). The integrability theorem says that if this demand function is preference-generated then it is associated with an expenditure function $e(p, u)$ that solves the differential equation:

$$\frac{\partial e(p, u)}{\partial p} = \alpha p + \delta e(p, u) + \gamma. \tag{2}$$

Note we ignore the ε. That is the sense in which it is interpreted as "noise", or something other than preferences. Hausman shows that the solution is in the class

$$e(p, u) = c \exp(\delta p) - \frac{1}{\delta}(\alpha p + \frac{\alpha}{\delta} + \gamma)$$

where c is a constant of integration that can depend on u. Since utility can be arbitrarily transformed, Hausman simply sets $c = u$. So the proposed solution is

$$e(p, u) = u \exp(\delta p) - \frac{1}{\delta}(\alpha p + \frac{\alpha}{\delta} + \gamma) \tag{3}$$

Checking this potential solution, note that (3) implies

$$\frac{\partial e(p, u)}{\partial p} = \delta u \exp(\delta p) - \frac{\alpha}{\delta} \tag{4}$$

and plugging into (2) you can see it works.

Is this a "proper" expenditure function? It will be if

$$\frac{\partial^2 e(p, u)}{\partial p^2} \leq 0$$

which will require

$$\delta^2 u \exp(\delta p) \leq 0.$$

Now from (3)

$$\begin{aligned}
u \exp(\delta p) &= e(p, u) + \frac{1}{\delta}(\alpha p + \frac{\alpha}{\delta} + \gamma) && (5)\\
&\Rightarrow \delta^2 u \exp(\delta p) = \delta(\delta y + \alpha p + \frac{\alpha}{\delta} + \gamma) \qquad (\textit{substituting } y = e) \\
&= \delta(x + \frac{\alpha}{\delta}) = \alpha + \delta x && (6)
\end{aligned}$$

Applying Slutsky's equation, notice that if we have the linear demand system (1) then

$$
\begin{aligned}
\frac{\partial x^c}{\partial p} &= \frac{\partial x}{\partial p} + \frac{\partial x}{\partial y} x \\
&= \alpha + \delta x
\end{aligned}
$$

So if the "Slutsky condition" $\frac{\partial x^c}{\partial p} \leq 0$ the expenditure function will be concave.