Economics 101A
(Lecture 6, Revised)

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Outline

1. Common Utility Functions

2. Utility Maximization with Lagrangeans

3. Utility maximization – tricky cases
1 Common utility functions

- Nicholson, Ch. 3, pp. 80–84

1. Cobb-Douglas preferences: \( u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha} \)
   
   - \( MRS = \frac{-\alpha x_1^{a-1} x_2^{1-\alpha} / (1-a) x_1^\alpha x_2^{-\alpha}}{-\alpha x_2^{1-\alpha} x_1^{\alpha-1}} = -\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} \) [REVISED]

2. Perfect substitutes: \( u(x_1, x_2) = \alpha x_1 + \beta x_2 \)
   
   - \( MRS = -\alpha/\beta \)
3. Perfect complements: \( u(x_1, x_2) = \min(\alpha x_1, \beta x_2) \)

- \( MRS \) discontinuous at \( x_2 = \frac{\alpha}{\beta} x_1 \)

4. Constant Elasticity of Substitution: \( u(x_1, x_2) = (\alpha x_1^\rho + \beta x_2^\rho)^{1/\rho} \)

- \( MRS = -\frac{\alpha}{\beta} \left( \frac{x_1}{x_2} \right)^{\rho - 1} \)

- if \( \rho = 1 \), then...

- if \( \rho = 0 \), then...

- if \( \rho \to -\infty \), then... [CORRECTION]
2 Utility Maximization

- Nicholson, Ch. 4, pp. 91–103

- $X = R_+^2$ (2 goods)

- Consumers: choose bundle $x = (x_1, x_2)$ in $X$ which yields highest utility.

- Constraint: income = $M$

- Price of good 1 = $p_1$, price of good 2 = $p_2$

- Bundle $x$ is feasible if $p_1x_1 + p_2x_2 \leq M$

- Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$

$$s.t. \ p_1x_1 + p_2x_2 \leq M$$

$$x_1 \geq 0, \ x_2 \geq 0$$
• Maximization subject to inequality. How do we solve that?

• Trick: $u$ strictly increasing in at least one dimension. ($\succeq$ strictly monotonic)

• Budget constraint always satisfied with equality

• Ignore temporarily $x_1 \geq 0, x_2 \geq 0$ and check afterwards that they are satisfied for $x_1^*$ and $x_2^*$. 
• Problem becomes

\[
\max_{x_1, x_2} u(x_1, x_2) \\
\text{s.t. } p_1 x_1 + p_2 x_2 - M = 0
\]

• \( L(x_1, x_2) = u(x_1, x_2) - \lambda (p_1 x_1 + p_2 x_2 = M) \)

• F.o.c.s:

\[
\begin{align*}
  u'_{x_i} - \lambda p_i & = 0 \text{ for } i = 1, 2 \\
p_1 x_1 + p_2 x_2 - M & = 0
\end{align*}
\]
• Moving the two terms across and dividing, we get:

\[ MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2} \]

• Graphical interpretation.
• Example with CES utility function.

\[
\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho}
\]
\[
s.t. \ p_1 x_1 + p_2 x_2 - M = 0
\]

• Lagrangean =

• F.o.c.:

• Special case: \( \rho = 0 \) (Cobb-Douglas)
3 Utility maximization – tricky cases

1. Non-convex preferences. Example:

- Second order conditions:

\[
H = \begin{pmatrix}
0 & -p_1 & -p_2 \\
-p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\
-p_2 & u''_{x_2,x_1} & u''_{x_2,x_2}
\end{pmatrix}
\]

\[
|H| = p_1 \left(-p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1}\right) \\
- p_2 \left(-p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1}\right) \\
= -p_1^2 u''_{x_2,x_2} + 2p_1p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1}
\]
2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

$$\max x_1^* (x_2 + 5)$$
$$s.t. \quad p_1 x_1 + p_2 x_2 = M$$

- In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?
3. Multiplicity of solutions. Example:

- Convex preferences that are not strictly convex
4. Example with CES utility function.

$$\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho}$$

$$s.t. \ p_1 x_1 + p_2 x_2 - M = 0$$

- With $\rho > 1$ the interior solution is a minimum!

- Draw indifference curves for $\rho = 1$ (boundary case) and $\rho = 2$

- Can also check using second order conditions