Economics 101A
(Lecture 16, Revised)

Stefano DellaVigna

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Outline

1. Addenda to isoquants

2. Cost Minimization: Summary

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4. Geometry of Cost Curves
1 Addenda to isoquants

- Production function $f(L, K)$

- When are isoquants convex? When $d^2 K/d^2 L > 0$

- Mathematically,
  $$
  \frac{dK}{dL}\big|_{\text{isoquant}} = -\frac{f'_L(L, K(L))}{f'_K(L, K(L))}
  $$

- What is
  $$
  \frac{d^2 K}{d^2 L}\big|_{\text{isoquant}} = ?
  $$

  Good exercise!
• The steps are as follows:

• $d^2K/d^2L$ is the second derivative with respect to $L$ of $dK/dL$. It follows $\left. \frac{d^2K}{d^2L} \right|_{isoquant} =$

$$
\left. \frac{\left( f_{L,L}'' (L, K) + f_{L,K}'' (L, K) \frac{\partial K(L)}{\partial L} \right) f'_K (L, K)}{(f'_K (K, L))^2} \right. \\
\left. \frac{\left( f_{K,L}'' (L, K) + f_{K,K}'' (L, K) \frac{\partial K(L)}{\partial L} \right) f'_L (L, K)}{(f'_K (K, L))^2} \right.
$$

• Substitute in

$$
\frac{\partial K (L)}{\partial L} = -\frac{f'_L (L, K (L))}{f'_K (L, K (L))}
$$

• Simplify and get
\[
\frac{d^2 K}{d^2 L} \bigg|_{\text{isoquant}} = \frac{-f''_{L,L}(L, K(L)) f'_K(L, K(L))}{\left(f'_K(K(L), L)\right)^2} - \frac{2 f''_{K,L}(L, K(L)) f'_L(L, K(L))}{\left(f'_K(K(L), L)\right)^2} + \frac{f'''_{K,K}(L, K(L)) \left(\frac{f'_L(L, K(L))}{f'_K(L, K(L))}\right)^2}{\left(f'_K(K(L), L)\right)^2} - \frac{f''_{K,L}(L, K(L)) \left(\frac{f'_L(L, K(L))}{f'_K(L, K(L))}\right)^2}{\left(f'_K(K(L), L)\right)^2}
\]

- Terms 1 and 3 are always positive.

- Term 2 is positive if \( f''_{K,L}(L, K(L)) \geq 0 \)

- Conclusion: \( f''_{K,L}(L, K(L)) \geq 0 \) is the only additional assumption we need to guarantee convex isoquants \( \left(\frac{d^2 K}{d^2 L} > 0\right) \)
2 Cost Minimization: Summary

• First stage. Firm’s objective function:

\[
\min_{L,K} wL + rK \quad \text{s.t.} \quad f(L, K) \geq y
\]

• Equality in constraint holds if:

1. \( w > 0, r > 0; \)

2. \( f \) strictly increasing in at least \( L \) or \( K. \)

• Counterexample if ass. 1 is not satisfied

• Counterexample if ass. 2 is not satisfied
• Second stage. Firm’s objective function:

$$\max_y py - c(w, r, y)$$

• First order condition:

$$p - c'_y (w, r, y^*) = 0$$

• Second order condition:

$$-c''_{y,y} (w, r, y^*) < 0$$

• For maximum, need increasing marginal cost curve.
3 Cost Minimization: Example

• [HEAVILY REVISED BELOW]

• Continue example above: \( y = f(L, K) = AK^\alpha L^\beta \)

• Cost minimization:

\[
\begin{align*}
\min & \quad wL + rK \\
\text{s.t.} & \quad AK^\alpha L^\beta = y
\end{align*}
\]

• Solutions:

  – Optimal amount of labor:

\[
L^*(r, w, y) = \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha+\beta}}
\]
- Optimal amount of capital:

\[
K^* (r, w, y) = \frac{w \alpha}{r \beta} \left( \frac{y}{A} \right)^{\frac{1}{\alpha+\beta}} \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha+\beta}} = (\frac{y}{A})^{\frac{1}{\alpha+\beta}} \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha+\beta}}
\]

- Check various comparative statics:

- \( \partial L^*/\partial A < 0 \) (technological progress and unemployment)

- \( \partial L^*/\partial y > 0 \) (more workers needed to produce more output)

- \( \partial L^*/\partial w < 0, \partial L^*/\partial r > 0 \) (substitute away from more expensive inputs)

- Parallel comparative statics for \( K^* \)
• Cost function

\[ c(w, r, y) = wL^*(r, w, y) + rK^*(r, w, y) = \]

\[ = (\frac{y}{A})^{\frac{1}{\alpha + \beta}} \left[ w \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha + \beta}} + \right. \]

\[ \left. + r \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha + \beta}} \right] \]

• Define \( B := w \left( \frac{w \alpha}{r \beta} \right)^{-\frac{\alpha}{\alpha + \beta}} + r \left( \frac{w \alpha}{r \beta} \right)^{\frac{\beta}{\alpha + \beta}} \)

• Cost-minimizing output choice:

\[ \max p y - B \left( \frac{y}{A} \right)^{\frac{1}{\alpha + \beta}} \]
• First order condition:

\[ p - \frac{1}{\alpha + \beta} \frac{B}{A} \left( \frac{y}{A} \right)^{\frac{1-(\alpha+\beta)}{\alpha+\beta}} = 0 \]

• Second order condition:

\[ -\frac{1}{\alpha + \beta} \frac{1 - (\alpha + \beta)}{\alpha + \beta} \frac{B}{A^2} \left( \frac{y}{A} \right)^{\frac{1-2(\alpha+\beta)}{\alpha+\beta}} \]

• When is the second order condition satisfied?
Solution:

\[ \alpha + \beta = 1 \text{ (CRS):} \]

* S.o.c. equal to 0

* Solution depends on \( p \)

* For \( p > \frac{1}{\alpha + \beta} \frac{B}{A} \), produce \( y^* \rightarrow \infty \)

* For \( p = \frac{1}{\alpha + \beta} \frac{B}{A} \), produce any \( y^* \in [0, \infty) \)

* For \( p < \frac{1}{\alpha + \beta} \frac{B}{A} \), produce \( y^* = 0 \)
- \( \alpha + \beta > 1 \) (IRS):
  
  * S.o.c. positive

  * Solution of f.o.c. is a minimum!

  * Solution is \( y^* \to \infty \).

  * Keep increasing production since higher production is associated with higher returns
- $\alpha + \beta < 1$ (DRS):
  * s.o.c. negative. OK!
  * Solution of f.o.c. is an interior optimum
  * This is the only "well-behaved" case under perfect competition
  * Here can define a supply function


4 Geometry of cost curves


- Marginal costs $MC = \partial c/\partial y \rightarrow$ Cost minimization

$$p = MC = \partial c(w, r, y)/\partial y$$

- Average costs $AC = c/y \rightarrow$ Does firm break even?

$$\pi = py - c(w, r, y) > 0 \text{ iff}$$

$$\pi/y = p - c(w, r, y)/y > 0 \text{ iff}$$

$$c(w, r, y)/y = AC < p$$

- Supply function. Portion of marginal cost $MC$ above average costs. (price equals marginal cost)
• Assume only 1 input (expenditure minimization is trivial)

• **Case 1.** Production function. $y = L^\alpha$

  - Cost function? (cost of input is $w$):
    $$c(w, y) = wL^*(w, y) = wy^{1/\alpha}$$

  - Marginal cost?
    $$\frac{\partial c(w, y)}{\partial y} = \frac{1}{\alpha}wy(1-\alpha)/\alpha$$

  - Average cost $c(w, y)/y$?
    $$\frac{c(w, y)}{y} = \frac{wy^{1/\alpha}}{y} = wy(1-\alpha)/\alpha$$
• **Case 1a.** $\alpha > 1$. Plot production function, total cost, average and marginal. Supply function?

• **Case 1b.** $\alpha = 1$. Plot production function, total cost, average and marginal. Supply function?

• **Case 1c.** $\alpha < 1$. Plot production function, total cost, average and marginal. Supply function?
• **Case 2.** *Non-convex technology.* Plot production function, total cost, average and marginal. Supply function?

• **Case 3.** *Technology with setup cost.* Plot production function, total cost, average and marginal. Supply function?