Outline

1. Intertemporal choice II

2. Altruism and charitable donations

3. Introduction to probability

4. (Expected Utility)
1 Intertemporal choice II

- Maximization problem:
  \[
  \max U(c_0) + \frac{1}{1 + \delta} U(c_1)
  \]
  \[
  s.t. \ c_0 + \frac{1}{1 + r} c_1 \leq M_0 + \frac{1}{1 + r} M_1
  \]

- Rewrite ratio of f.o.c.s and budget constraint as
  \[
  U'(c_0) - \frac{1 + r}{1 + \delta} U'(M_1 + (M_0 - c_0)(1 + r)) = 0
  \]
• Comparative statics with respect to interest rate $r$

• Apply implicit function theorem:

$$\frac{\partial c_0^*(r, M)}{\partial r} = -\frac{\frac{1}{1+\delta} U''(c_1)}{U'''(c_0) - \frac{1+r}{1+\delta} U'''(c_1) * (-1+r)} \frac{-\frac{1+r}{1+\delta} U''(c_1) * (M_0 - c_0)}{U'''(c_0) - \frac{1+r}{1+\delta} U'''(c_1) * (-1+r)}$$

• Denominator is always negative

• Numerator: First term is negative (substitution effect)

• Second term is income effect:
  
  - positive if $M_0 > c_0$
  
  - negative if $M_0 < c_0$. 

2 Altruism and Charitable Donations

• Maximize utility = satisfy self-interest?

• No, not necessarily

• 2-person economy:
  – Mark has income $M_M$ and consumes $c_M$
  – Wendy has income $M_W$ and consumes $c_W$

• One good: $c$, with price $p = 1$
• Utility function: $u(c)$, with $u' > 0$, $u'' < 0$

• Wendy is altruistic: she maximizes $u(c_W) + \alpha u(c_M)$ with $\alpha > 0$

• Mark simply maximizes $u(c_M)$

• Wendy can give a donation of income $D$ to Mark.
• Wendy computes the utility of Mark as a function of the donation $D$

• Mark maximizes

$$\max_{c_M} u(c_M)$$

$$s.t. \ c_M \leq M_M + D$$

• Solution: $c_M^* = M_M + D$

• Wendy maximizes

$$\max_{c_M, D} u(c_W) + \alpha u(M_M + D)$$

$$s.t. \ c_W \leq M_W - D$$
• Rewrite as:

$$\max_{D} u(M_W - D) + \alpha u(M_M + D)$$

• First order condition:

$$-u'(M_W - D^*) + \alpha u'(M_M + D^*) = 0$$

• Second order conditions:

$$u''(M_W - D^*) + \alpha u''(M_M + D^*) < 0$$
• Assume $\alpha = 1$.

  – Solution?

  – $u'(M_W - D) = u'(M_M + D^*)$

  – $M_W - D^* = M_M + D^*$ or $D^* = (M_W - M_M)/2$

  – Transfer money so as to equate incomes!

  – Careful: $D < 0$ (negative donation!) if $M_M > M_W$

• Corrected maximization:

  $\max_D u(M_W - D) + \alpha u(M_M + D)$

  s.t. $D \geq 0$

• Solution ($\alpha = 1$):

  $D^* = \begin{cases} 
  (M_W - M_M)/2 & \text{if } M_W - M_M > 0 \\
  0 & \text{otherwise}
  \end{cases}$
• Assume interior solution. \((D^* > 0)\)

• Comparative statics 1 (altruism):
\[
\frac{\partial D^*}{\partial \alpha} = -\frac{u'(M_M + D^*)}{u''(M_W - D^*) + \alpha u''(M_M + D^*)} > 0
\]

• Comparative statics 2 (income of donor):
\[
\frac{\partial D^*}{\partial M_W} = -\frac{-u''(M_W + D^*)}{u''(M_W - D^*) + \alpha u''(M_M + D^*)} > 0
\]

• Comparative statics 3 (income of recipient):
\[
\frac{\partial D^*}{\partial M_M} = -\frac{\alpha u''(M_M + D^*)}{u''(M_W - D^*) + \alpha u''(M_M + D^*)} < 0
\]
3 Introduction to Probability

- So far deterministic world:
  - income given, known $M$
  - interest rate known $r$

- But some variables are unknown at time of decision:
  - future income $M_1$?
  - future interest rate $r_1$?

- Generalize framework to allow for uncertainty
  - Events that are truly unpredictable (weather)
  - Event that are very hard to predict (future income)
• Probability is the language of uncertainty

• Example:
  
  – Income $M_1$ at $t = 1$ depends on state of the economy
  
  – Recession ($M_1 = 20$), Slow growth ($M_2 = 25$), Boom ($M_3 = 30$)

  – Three probabilities: $p_1$, $p_2$, $p_3$
  
  – $p_1 = P(M_1) = P($recession$)$

• Properties:
  
  – $0 \leq p_i \leq 1$

  – $p_1 + p_2 + p_3 = 1$
• Mean income:  
$$EM = \sum_{i=1}^{3} p_i M_i$$

• If \((p_1, p_2, p_3) = (1/3, 1/3, 1/3)\),

$$EM = \frac{1}{3} 20 + \frac{1}{3} 25 + \frac{1}{3} 30 = \frac{75}{3} = 25$$

• Variance of income:  
$$V(M) = \sum_{i=1}^{3} p_i (M_i - EM)^2$$

• If \((p_1, p_2, p_3) = (1/3, 1/3, 1/3)\),

$$V(M) = \frac{1}{3} (20 - 25)^2 + \frac{1}{3} (25 - 25)^2 + \frac{1}{3} (30 - 25)^2$$

$$= \frac{1}{3} 5^2 + \frac{1}{3} 5^2 = \frac{2}{3} \times 25$$

• Mean and variance if \((p_1, p_2, p_3) = (1/4, 1/2, 1/4)\)?
4 Expected Utility

- Nicholson, Ch. 18, pp. 533–541 [OLD: Ch. 8, pp. 198–206]

- Consumer at time 0 asks: what is utility in time 1?

- At $t = 1$ consumer maximizes

  \[
  \max U(c^1) \\
  s.t. \ c_i^1 \leq M_i^1 + (1 + r) (M^0 - c^0)
  \]

  with $i = 1, 2, 3$.

- What is utility at optimum at $t = 1$ if $U' > 0$?

- Assume for now $M^0 - c^0 = 0$

- Utility $U\left(M_i^1\right)$

- This is uncertain, depends on which $i$ is realized!
• How do we evaluate future uncertain utility?

• Expected utility

\[
EU = \sum_{i=1}^{3} p_i U \left( M_i \right)
\]

• In example:

\[
EU = \frac{1}{3} U(20) + \frac{1}{3} U(25) + \frac{1}{3} U(30)
\]

• Compare with \( U(EC) = U(25) \).

• Agents prefer riskless outcome \( EM \) to uncertain outcome \( M \) if

\[
\frac{1}{3} U(20) + \frac{1}{3} U(25) + \frac{1}{3} U(30) < U(25) \quad \text{or} \quad \frac{1}{2} U(20) + \frac{1}{2} U(30) < \frac{2}{3} U(25) \quad \text{or} \quad \frac{1}{2} U(20) + \frac{1}{2} U(30) < U(25)
\]
• Picture
• Depends on sign of $U''$, on concavity/convexity

• Three cases:

  - $U''(x) = 0$ for all $x$. (linearity of $U$)
    * $U(x) = a + bx$
    * $1/2U(20) + 1/2U(30) = U(25)$

  - $U''(x) < 0$ for all $x$. (concavity of $U$)
    * $1/2U(20) + 1/2U(30) < U(25)$

  - $U''(x) > 0$ for all $x$. (convexity of $U$)
    * $1/2U(20) + 1/2U(30) > U(25)$
• If $U'''(x) = 0$ (linearity), consumer is indifferent to uncertainty

• If $U'''(x) < 0$ (concavity), consumer dislikes uncertainty

• If $U'''(x) > 0$ (convexity), consumer likes uncertainty

• Do consumers like uncertainty?

• Do you like uncertainty?
• **Theorem. (Jensen’s inequality)** If a function \( f(x) \) is concave, the following inequality holds:

\[
f(Ex) \geq Ef(x)
\]

where \( E \) indicates expectation. If \( f \) is strictly concave, we obtain

\[
f(Ex) > Ef(x)
\]

• Apply to utility function \( U \).

• Individuals dislike uncertainty:

\[
U(Ex) \geq EU(x)
\]

• Jensen’s inequality then implies \( U \) concave (\( U'' \leq 0 \))

• Relate to diminishing marginal utility of income